CHAPTER 7

FURTHER CONCLUSIONS AND SOME PROBLEMS FOR FUTURE RESEARCH

This last chapter differs enough from the previous six to warrant a brief introduction. In the first six chapters a theoretical structure was introduced and several basic theorems lying close to the definitions were proven. Many examples dealing with the simplest generators, the linear ones, were discussed in detail in order to illustrate the theory and to show its potential usefulness. Few problems were left to the reader.

In the present chapter we will not be primarily concerned with introducing theory but will instead be considering flows generated by algebraic curves of higher degree as well as flows generated by time dependent generators and time dependent local rates. Generators of degree two and higher lead to flows too complicated to discuss in complete detail here so several routine

exercises will be left to the reader. Some more difficult problems of a theoretical nature will also be posed in the last section.

A. Quadratic and Higher Order Generators

One of the most interesting features of systems of nonlinear differential equations is their ability to produce limit cycles, for example van der Pol's equation, which has been studied extensively in the literature and is also discussed in some contemporary texts on differential equations, such as Ross [1]. The master equation will not produce limit cycles if we only use linear generators. A combination of a circle and a real line or a circle and a pair of complex lines will produce a limit cycle, but before discussing limit cycles per se let us first consider quadratic and higher order generators in a systematic way.

There are three distinct quadratic generators in E^{2+}, the parabola, the ellipse and the hyperbola. Each of these may be real or complex. We should consider both class I and class II flows generated by quadratic curves. Let us start by considering the class I flows generated by real parabolas, ellipses and hyperbolas.

Without loss of generality let the parabola be $P=x^2-y$, then $\dot{x}=-A_1$, $\dot{y}=-2A_1x$. This flow was discussed in

Chapter Two (see Figure 2.4). To study the flow generated by an ellipse let $P=x^2+y^2-1$. We have $\dot{x}=2A_1y$, $\dot{y}=-2A_1x$. This flow is the same as that generated by a pair of complex lines centered at the origin, where the local rate is $2iA_1$ (see Figure 2.1.1). Finally let $P=x^2-y^2-1$ to get the class I flow generated by a real hyperbola. Then $\dot{x}=-2A_1y$, $\dot{y}=-2A_1x$. In this case the flow is the same as that generated by two real lines, $L_1=x-y$ and $L_2=x+y$, where $A_1=-A_2$ (see Figure 2.1.3).

Though none of these three flows are in themselves very interesting there are two features worth noting. First the quadratic generators, when used by themselves, are not separatrices. We will see that when quadratic generators are used in conjunction with other generators they generally become separatrices. Secondly we see that for a given flow the generators are not unique. More will be said concerning this second point in section C.

Let us now consider the class II flow generated by a real parabola. The system is

$$\begin{aligned}\dot{x} &= -A_1 + A_\infty(x^2-y)\\ \dot{y} &= -2A_1x - B_\infty(x^2-y).\end{aligned}$$

Furthermore $C(u,v,o)=-u^2(B_\infty u+A_\infty v)$. If $A_\infty\neq 0$ we lose no generality by setting $B_\infty=0$. The class II critical points at infinity occur at $(\pm 1,0)$. Using either Theorem 1.10

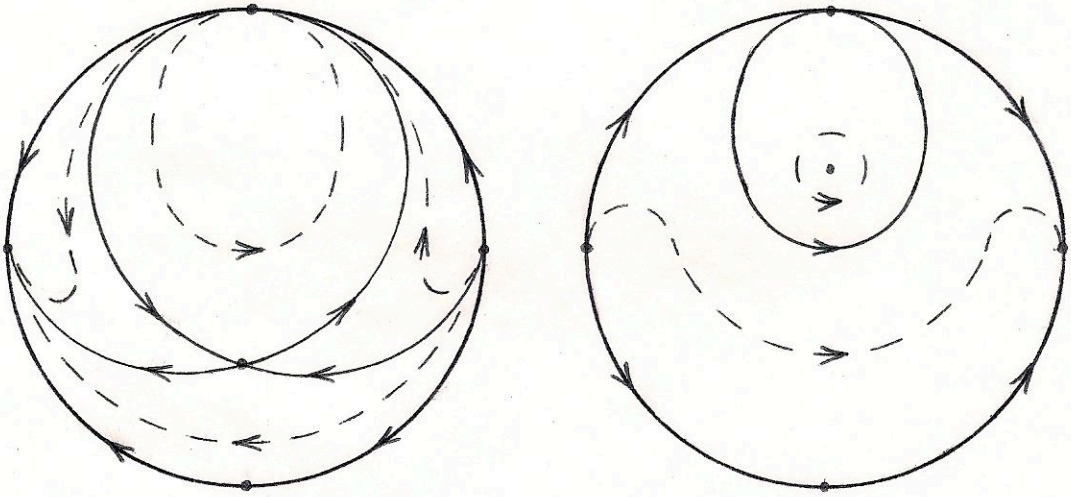
or Lemma 1.4 we find

$$d\eta/d\sigma\big|_{(1,0)} = -A_\infty\eta - 2A_1\xi + \eta\xi(A_\infty\eta + A_1\xi)$$

$$d\xi/d\sigma\big|_{(1,0)} = -A_\infty\xi + \xi^2(A_\infty\eta + A_1\xi).$$

The flow near $(\pm 1,0)$ is like that near a one-tangent node.

The point $(0,-A_1/A_\infty)$, in E^2, is a spontaneous critical point. The linearized flow at $(0,-A_1/A_\infty)$ is $\dot{\underline{x}}=-A_\infty\underline{y}$, $\dot{\underline{y}}=-2A_1\underline{x}$ so we have a saddle if $A_1A_\infty>0$ and a center if $A_1A_\infty<0$. See Figure 7.1.



7.1.1: $B_\infty=0$, $A_1<0$, $A_1A_\infty>0$

7.1.2: $B_\infty=0$, $A_1<0$, $A_1A_\infty<0$

Figure 7.1: Class II flows generated by $P=x^2-y$ with $B_\infty=0$.

Now consider the $A_\infty=0$ case. Then $\dot{x}=-A_1$ and $\dot{y}=-2A_1x-B_\infty(x^2-y)$. We get no spontaneous critical points in E^2 and the only critical points at infinity are located at $(0,\pm1)$. See Figure 7.2.

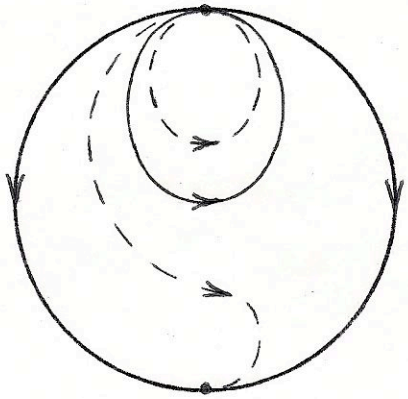


Figure 7.2: The third class II flow generated by $P=x^2-y$ with $A_\infty=0$.

Exercise 1: Find all of the distinct class II flows in E^{2+} generated by an ellipse. Do the same for the hyperbola.

Let us now consider complex quadratic generators. We will first prove a lemma similar to Lemma 5.1.

Lemma 7.1: Suppose that P is a complex quadratic curve. If X and Y, as given by the master equation, are real then $A_2=\overline{A}_1$.

Proof: Without loss of generality we may write every complex quadratic curve either in the form $P=x^2+by$, $\mathrm{Im}(b)\neq 0$, or $P=ax^2+by^2-1$, where a and b are not both real and $|a|^2\neq 0$, $|b|^2\neq 0$. For a class II flow A_∞ is always the coefficient of the term of highest degree so A_∞ is real. Now for the parabola we have

$$X-\overline{X} = 2i\mathrm{Im}((A_1-\overline{A}_2)b)x^2 + 2i\mathrm{Im}(A_1-\overline{A}_2)|b|^2y = 0,$$

for all values of x and y. Since $|b|^2\neq 0$ we get $\mathrm{Im}(A_1-\overline{A}_2)=0$. Then

$$\mathrm{Im}((A_1-\overline{A}_2)b) = \mathrm{Im}(A_1-\overline{A}_2)\mathrm{Re}(b)+\mathrm{Im}(b)\mathrm{Re}(A_1-\overline{A}_2) = 0$$

implies $\mathrm{Re}(A_1-\overline{A}_2)=0$. Hence $A_2=\overline{A}_1$.

For the ellipse or hyperbola we find that

$$X-\overline{X} = 4i\mathrm{Im}((A_1-\overline{A}_2)b\overline{a})x^2 + 4i\mathrm{Im}(A_1-\overline{A}_2)|b|^2y^2 - 4i\mathrm{Im}((A_1-\overline{A}_2)b) = 0,$$

for all values of x and y. Since $|b|^2\neq 0$ we get $\mathrm{Im}(A_1-\overline{A}_2)=0$. Now $\mathrm{Im}((A_1-\overline{A}_2)b)=0$ implies $\mathrm{Re}(A_1-\overline{A}_2)=0$, provided $\mathrm{Im}(b)\neq 0$. If $\mathrm{Im}(b)=0$ then $\mathrm{Im}((A_1-\overline{A}_2)b\overline{a})=0$ implies $\mathrm{Re}(A_1-\overline{A}_2)=0$, since $\mathrm{Im}(\overline{a})\neq 0$. Hence $A_2=\overline{A}_1$.

Q.E.D.

As a particular example consider the class I flow generated by $P_1=x^2+iy$ and $P_2=x^2-iy$. The local rate on P_1 is related to the global rate by $\Lambda_1=-A_1$. Let $\Lambda_1=\mu+i\omega$. Then we have

$$\dot{x} = -2(\mu y-\omega x^2)$$
$$\dot{y} = 4x(\omega y+\mu x^2).$$

If $\mu\neq 0$ the only finite critical point is at $(0,0)$. Furthermore $C(u,v,0)=4\mu u^4$ so if $\mu\neq 0$ the only critical points at infinity occur at $(0,\pm 1)$. If $\mu=0$ then every point at infinity is a critical point and the y axis becomes a line of critical points. See Figure 7.3.1. If $\omega=0$ the system is $\dot{x}=-2\mu y$, $\dot{y}=4\mu x^3$. By Poincare's Theorem (5.2) we see that the origin is then a center. See Figure 7.3.2.

To study the flow near u=0 we must first rotate by $\Theta=\pi/2$. We find

$$X_R = -4y(\omega x+\mu y^2)$$
$$Y_R = 2(\mu x-\omega y^2)$$

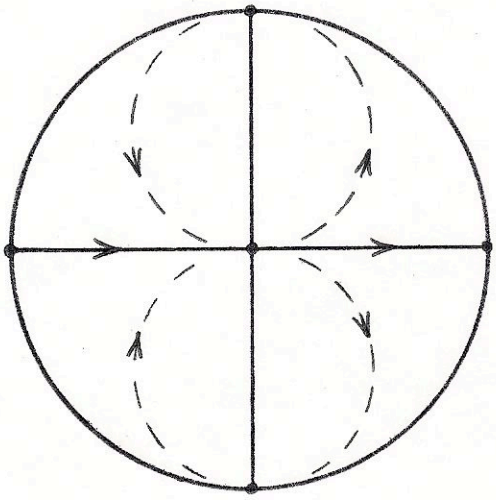
and

$$\dot{\eta}\big|_{(0,1)} = 2\mu\xi^2+2\eta^2(\omega\xi+2\mu\eta^2)$$
$$\dot{\xi}\big|_{(0,1)} = 4\eta\xi(\mu\eta^2+\omega\xi)$$

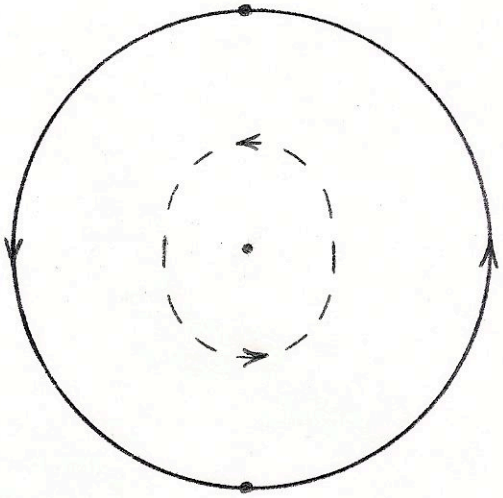
so

$$(\eta\dot{\xi}-\xi\dot{\eta})\big|_{(0,1)} = 2\xi^2(\omega\eta^2-\mu\xi).$$

The flows for which $\mu\omega\neq 0$ are equivalent to the flow for which $\omega=0<\mu$.



7.3.1: $\mu=0<\omega$
$\dot{x}=x^2$, $\dot{y}=2xy$



7.3.2: $\omega=0<\mu$
$\dot{x}=-y$, $\dot{y}=2x^3$

Figure 7.3: The two types of class I flows generated by a complex parabola and its conjugate.

Exercise 2: Find all of the distinct class I flows in E^{2+} generated by $P_1=ax^2+by^2-1$, and $P_2=\bar{a}x^2+\bar{b}y^2-1$, where a and b are not both real and $|a|^2\neq 0$, $|b|^2\neq 0$.

Exercise 3: Find all of the distinct class II flows in E^{2+} generated by a complex quadratic curve and its conjugate.

We will now consider some class I flows generated by combinations of quadratic and linear generators. First suppose $P_1=y-x^2$ and $P_2=y+c$. We have

$$\dot{x} = cA_2+(A_1+A_2)y-A_1x^2 \qquad 7.1$$

$$\dot{y} = 2cA_2x+2A_2xy \qquad 7.2$$

If $c\leq 0$ we get natural critical points at $(\pm\sqrt{-c},-c)$. There is a spontaneous critical point at $(0,-cA_2/(A_1+A_2))$. Notice that when $c=0$ the spontaneous and natural critical points are all at $(0,0)$. We find that $C(u,v,0)=(2A_1+A_2)u^2v$ so there are four critical points at infinity, unless $2A_1+A_2=0$. In the latter case every point at infinity is a critical point.

Calculating the local rates in terms of the global rates gives $\Lambda_1(0,-c)=-cA_1$ and $\Lambda_2(0,0)=cA_2$, when $c\neq 0$; and $\Lambda_1(0,0)=-A_1$ and $\Lambda_2(1,1)=\sqrt{5}A_2$, when $c=0$. In the latter case $\Lambda_2(0,0)=0$ since P_2 is then L_1. The equations may now be written in terms of the Λ_i's rather than the A_i's.

Exercise 4: Show that the system given by equations 7.1 and 7.2 generates nine topologically distinct flows in E^{2+} given by $\Lambda_1=1$, with $c=\pm 1$ and $\Lambda_2<0$ or $0<\Lambda_2<1$ or $\Lambda_2>1$; and by $\Lambda_1=1$, $c=0$ with $\Lambda_2(1,1)<2\sqrt{5}$ or $\Lambda_2(1,1)=2\sqrt{5}$ or $\Lambda_2(1,1)>2\sqrt{5}$.

Exercise 5: Investigate the class I flows generated by two real parabolas.

As a second example consider a class I flow generated by a circle and a complex line on the circle. Let $P_1=x+iy$, $P_2=x-iy$ and $P_3=(x-1)^2+y^2-1$. We have

$$\dot{x} = A_1 iP_2P_3+A_2(-i)P_1P_3+A_3 2y|P_1|^2 \qquad 7.3$$

$$\dot{y} = -A_1P_2P_3-A_2P_1P_3-A_3 2(x-1)|P_1|^2 . \qquad 7.4$$

Exercise 6: Show that if $\dot{x}$, as given in equation 7.3, is real then A_3 is real and $A_2=\bar{A}_1$.

Calculating the local rates in terms of the global rates gives $\Lambda_1(0,0)=2\sqrt{2}\ A_1$ and $\Lambda_3(2,0)=-8A_3$. If we let $\Lambda_1(0,0)=2\sqrt{2}\ a$ and $\Lambda_3(2,0)=-8$ then $A_1=A_2=a$, $A_3=1$ and the flow reduces to

$$\dot{x} = 4ay(x^2-x+y^2) \qquad 7.5$$

$$\dot{y} = 2(2a+1)x^2+2y^2-2(1+a)x(x^2+y^2). \qquad 7.6$$

We have spontaneous critical points at $((2a+1)/(a+1),0)$ and $(1/(1-a),\pm\sqrt{-a}/(1-a))$. By considering $C(u,v,0)=-2(2a+(1-a)u^2)$ we see that the critical points at infinity are located at $(\pm\sqrt{2a/(a-1)}\ ,\pm\sqrt{(1+a)/(1-a)})$ and $(\pm\sqrt{2a/(a-1)}\ ,\ \mp\sqrt{(1+a)/(1-a)})$.

Exercise 7: Show that equations 7.5 and 7.6 generate seven distinct flows in E^{2+} and that the points of bifurcation occur at a=-1, -1/2 and 0.

By placing the pair of complex lines inside the circle we may force the circle to become a limit cycle. For example let $P_1=x+iy$, $P_2=x-iy$ and $P_3=x^2+y^2-1$ with $A_1=(1+i)/2$, $A_2=\overline{A}_1$ and $A_3=-1/2$. Then

$$\dot{x} = x-y-x(x^2+y^2) \qquad 7.7$$

$$\dot{y} = x+y-y(x^2+y^2) \qquad 7.8$$

or $\dot{r}=-r(r^2-1)$ and $\dot{\theta}=1$. The local rates are $\Lambda_1(0,0)=1-i$ and $\Lambda_3(0,1)=-1$. From the polar form it is easy to see

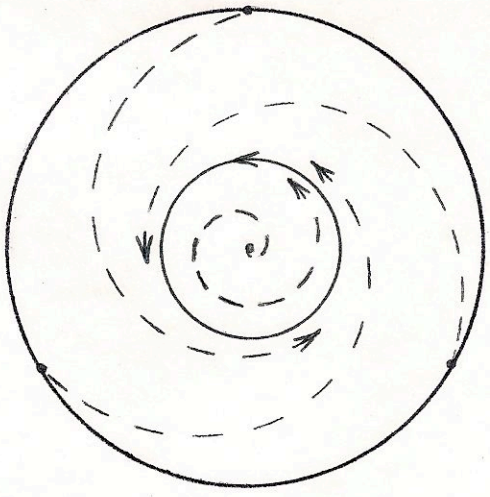


Figure 7.4: A simple example of a limit cycle.

that r=1 is a limit cycle. The only finite critical point is at (0,0) and every point at infinity is a critical point. See Figure 7.4. The flow given in equations 7.7 and 7.8 can be generalized as indicated in the next exercise.

Exercise 8: Show that the class I flow generated by $I=x+iy$, $\bar{I}=x-iy$ and n circles centered at the origin has n limit cycles if $\mathrm{Re}(\Lambda(0,0))\neq 0$.

The example given in equations 7.7 and 7.8 is cubic. It is possible to produce a limit cycle with only a quadratic flow. Consider the extension flow generated by the circle, x^2+y^2-1, and the straight line, $x+y+2$, where the circle is a primary generator and the straight line is a secondary generator. A particular example of such a flow is given by

$$\dot{x} = y(x+y+2)+(x^2+y^2-1) \qquad 7.9$$

$$\dot{y} = -x(x+y+2) \ . \qquad 7.10$$

The only finite critical points occur at $(0,(-1\pm\sqrt{3})/2)$. The only critical points at infinity are at $(\pm 2/\sqrt{5},\mp 1/\sqrt{5})$. Verifying that the circle is actually a limit cycle is nontrivial (see Chin Yuan-Shun [1], page 937). The flow corresponding to equations 7.9 and 7.10, as determined with the aid of the computer, is shown in Figure 7.5.

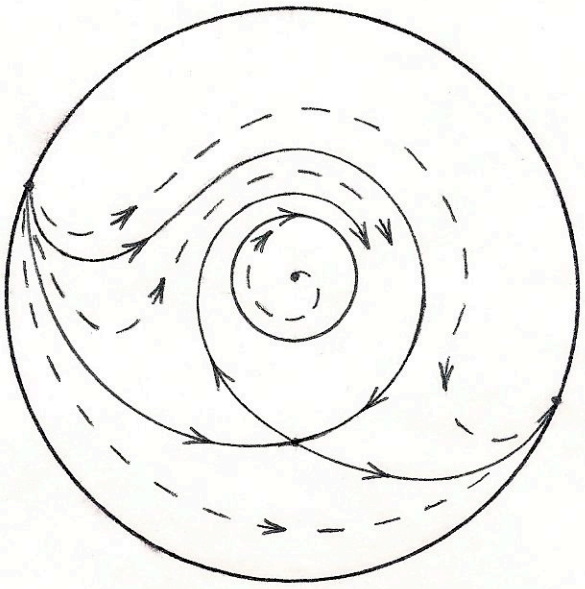


Figure 7.5: A limit cycle generated by a quadratic flow.

Let us take a brief detour at this point to discuss limit cycles in connection with the second part of Hilbert's famous sixteenth problem. The problem is to find the maximum number and relative positions of the limit cycles generated by the system $\dot{x}=X(x,y)$, $\dot{y}=Y(x,y)$ where X and Y are rational integral functions of degree n. (See Hilbert [1], page 464.) It is interesting to note that of the twenty two problems posed by Hilbert in 1900 which admit of mathematical solution (problem six, which deals with axiomatizing physics, seems to fall beyond the scope of mathematics) only problems twelve and

sixteen remain unsolved. It should also be noted that the second part of problem sixteen is unsolved even for the simplest nontrivial case, namely the case where X and Y are algebraic functions of degree two. Bautin [1] has solved the quadratic case locally by showing that the maximum number of limit cycles which can be made to appear near a weak focus is three. He has also produced systems with two and three limit cycles around a common critical point. Landis and Petrovskii [1] claim to have shown that three is also the maximum number globally but the validity of their proof is somewhat in doubt (see Coppel [1], page 295). In any event no quadratic systems with more than three limit cycles have as yet been found. If three is the maximum number of limit cycles then it is known that there are five possible distinct types of flows. The first three, which have already been discussed, are those with one critical point in E^2. We now turn our attention to the generation of the latter two types.

Tung Chin-Chu [1] has shown how to produce quadratic flows with two critical points in E^2 and two or three limit cycles. He makes use of the rotated vector fields introduced by G. F. D. Duff [1] to perturb a quadratic flow with two weak foci. We will show how flows equivalent to those of Tung Chin-Chu can be obtained using the master

equation, but first consider two lemmas.

Lemma 7.2: The rotated vector field, in the sense of G. F. D. Duff, given by

$$\dot{x} = X\cos\alpha - Y\sin\alpha$$
$$\dot{y} = X\sin\alpha + Y\cos\alpha,$$

where $\sin\alpha\cos\alpha \neq 0$, is equivalent to a sum flow if (X,Y) is generated by the master equation.

Proof: $(X\cos\alpha, Y\sin\alpha)$ is time-equivalent to (X,Y). Furthermore (-Y,X) is space-equivalent to (X,Y) and $(-Y\sin\alpha, X\sin\alpha)$ is time-equivalent to (-Y,X). Now add $(X\cos\alpha, Y\cos\alpha)$ and $(-Y\sin\alpha, X\sin\alpha)$ to get the desired conclusion.

Q.E.D.

Lemma 7.3: Every quadratic flow is a sum flow.

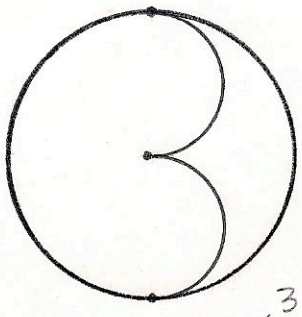
Proof: The sum flow given in Lemma 4.2 when added to the class I flow generated by $P = \sum_{j=1}^{3} c_j x^j + d_j y^j$ will produce any desired quadratic flow.

Q.E.D.

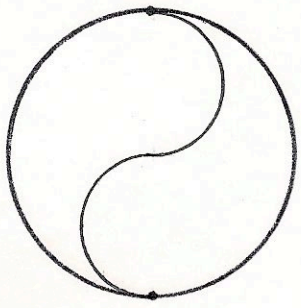
We see that the flows given by Tung Chin-Chu can either be thought of as sum flows directly, by Lemma 7.3,

or as perturbed sum flows, in the sense of Duff, by Lemma 7.2. Whether or not they are equivalent to either class I or class II flows is a difficult question for further consideration.

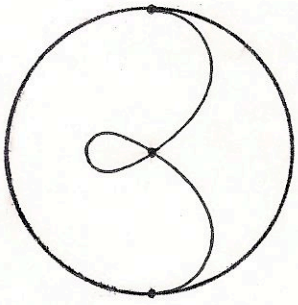
We return now to the main topic under consideration in this section, which is a study of the flows produced by higher order generators. There are at least eight distinct cubic generators in E^{2+}. See Figure 7.6 for particular examples.



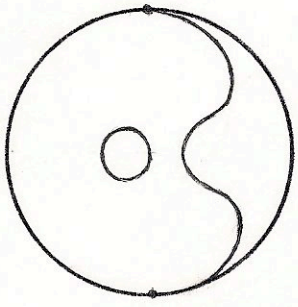
7.6.1: $y^2-x^3=0$



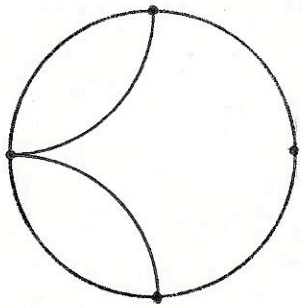
7.6.2: $y-x^3=0$



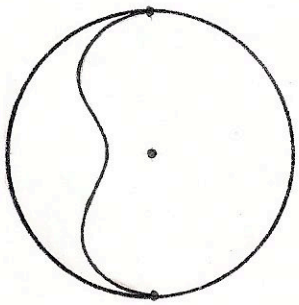
7.6.3: $y^2-x^3-x^2=0$



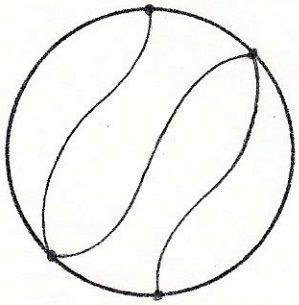
7.6.4: $y^2-x^3+x=0$



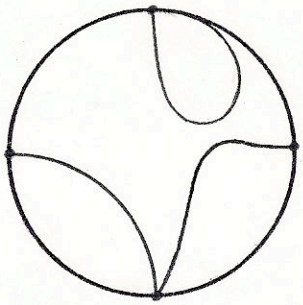
7.6.5: $xy^2+1=0$



7.6.6: $x^3+x^2+y^2=0$



7.6.7: $x^3-2x^2y+xy^2+x^2-xy-2x+y=0$



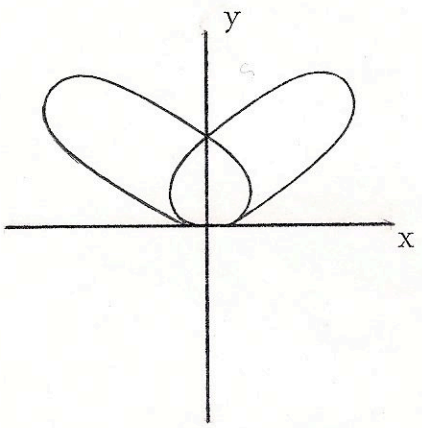
7.6.8: $yx(x-1)-2x+3=0$

Figure 7.6: Eight distinct cubic curves in E^{2+}.

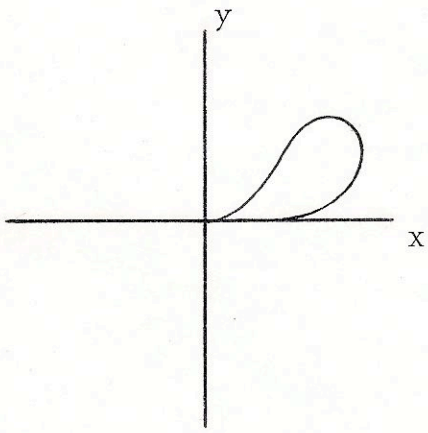
Cubic generators are seen to possess features not present in quadratic generators. For example the curve shown in Figure 7.6.1 has a cusp at the origin, while the origin is an ordinary double point for the curve in Figure 7.6.3. In Figure 7.6.6 the origin is an isolated point with tangents $x \pm iy$. Figures 7.6.7 and 7.6.8 show curves with

three branches in E^2.

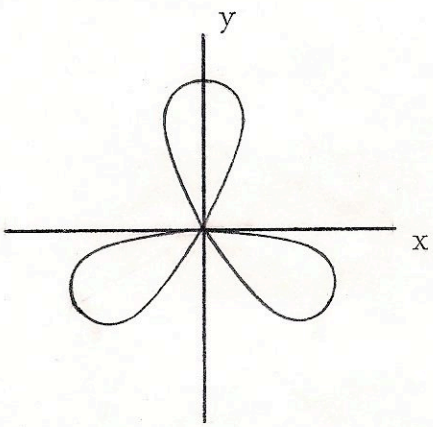
Algebraic curves of higher degree can have even more complicated features. For example see Figure 7.7.



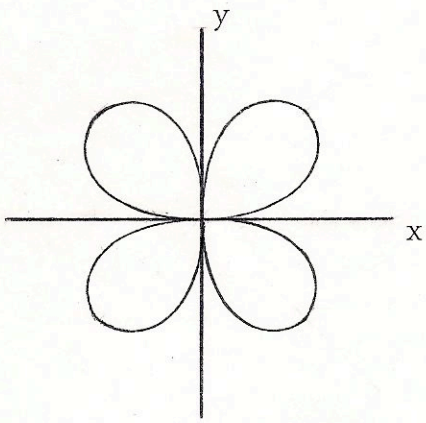
7.7.1: $2x^4-3x^2y+y^2-2y^3+y^4=0$



7.7.2: $x^4+x^2y^2-2x^2y-xy^2+y^2=0$



7.7.3: $(x^2+y^2)^2+3x^2y-y^3=0$



7.7.4: $(x^2+y^2)^3-4x^2y^2=0$

Figure 7.7: Algebraic curves of degree four and higher in E^2.

The topological classification of higher order algebraic curves is still an open problem. In fact it is equivalent to the first part of Hilbert's sixteenth problem.

Exercise 9: Find the distinct class I flows in E^{2+} generated by the cubic curves.

In the first section of Chapter Two the possibility of formulating a classification scheme for quadratic flows based on the use of the master equation was raised. It is now possible to suggest such a scheme. We may use the following types of generators to get a quadratic flow.

Class I: a) Three linear generators.
- i) all real
- ii) one real and a complex conjugate pair

b) One linear and one quadratic generator.

c) One cubic generator.

Class II: a) Two linear generators.
- i) both real
- ii) a complex conjugate pair

b) One quadratic generator.

Class III: a) Extension flows.

b) Power flows.

c) Sum flows.

Using such a scheme it is possible to set lower bounds on the number of topologically distinct quadratic flows in E^{2+}. For example there are 64 class I flows of type a), some of which have been discussed in detail in previous chapters. See Figures 2.9.1-2.9.13, 3.2.1-3.2.12, 6.2.1-6.2.16 and 6.4.1-6.4.5 for example. There are at least 9 and probably 27 class I flows of type b). See Exercise 4. The class I flows of type c) remain to be studied. See Exercise 9. There are 15 class II flows of type a). See Theorem 3.1 and Figures 2.16.1-2.16.10 and 5.1.1-5.1.2. There are at least 3 and probably 5 class II flows of type b). See Figures 7.1.1, 7.1.2 and 7.2 as well as Exercise 1. It seems possible that a complete classification of the class I and class II quadratic flows could be carried out.

When it comes to the class III quadratic flows the classification problem becomes a good deal more difficult. Counting the extension and power flows may be possible but counting the sum flows might be nearly impossible. It was conjectured that there could be as many as 270 sum flows of the form $\dot{x}=x(a_0+a_1x+a_2y)$, $\dot{y}=y(b_0+b_1x+b_2y)$. Of course since every quadratic flow may be written as a sum flow, by Lemma 7.3, there will be flows which may occur under several different types. For example $\dot{x}=x^2$, $\dot{y}=y^2$ is

a power flow, a sum flow and a class I flow of type a)i.

In closing this section it should be noted that a solution to the general enumeration and classification problem for algebraic flows would be equivalent to a solution to the second part of Hilbert's sixteenth problem. Furthermore a solution to the second part of the problem will probably necessitate a solution to the first part as well.

B. Further Extensions and Applications of the Theory

In this section we will discuss three different kinds of extensions of the theory. These extensions are: first, to nonautonomous flows; second, to nonalgebraic flows, and finally to flows in E^{2n} where $n>1$. Examples which illustrate the usefullness of such extensions are also discussed.

Nonautonomous flows can be arrived at in two distinct ways. First it is possible to introduce t into the right hand side of the master equation either through the local or global rates or through the generators. If the local rates are allowed to become functions of time we can keep the generators independent of time by allowing the corresponding global rates to carry the time dependence. On the other hand if we allow a generator to become time

dependent it may not always be possible to keep the local and global rates stationary. For example consider the class I flow generated by $L_1=y$ and $L_2(t) = -x\sin t+y\cos t$ which we viewed as a limit flow in Chapter Three. It was found that $\Lambda_1(0,0)=-A_1\sin t$ and $\Lambda_2(0,0)=A_2\sin t$. Not all of Λ_1, Λ_2, A_1 and A_2 can be constants with respect to t.

It is sometimes possible, however, to keep the local rates independent of time by evaluating them at time dependent points. For example consider the class I flow generated by $L_1=y$ and $L_2(t)=x-vt$. We find that $\Lambda_1(vt,0)=A_1$ and $\Lambda_2(vt,0)=A_2$. Hence if the global rates are constant, so are the moving local rates.

A second way of producing nonautonomous flows is by means of generalized sum flows. For example a simple model which simulates the development in the northern hemisphere of the weather phenomenon known to meteorologists as a cyclonic flow, more commonly known as a low pressure center, can be constructed by adding two linear flows as follows. Let $\vec{F}_1$ be the class I flow generated by $L_1=y+1$ and $L_2=y-1$ with $\Lambda_1(0,-1)=1$ and $\Lambda_2(0,1)=-1$. We get $X_1=-y$, $Y_1=0$. Let $\vec{F}_2$ be the class I flow generated by $I=x+iy$ and $\bar{I}$ with $\Lambda(0,0)=\mu+i\omega$. We find $X_2=\mu x+\omega y$, $Y_2=-\omega x+\mu y$. Suppose that

$$\vec{F}(t) = c_1(t)s_1(x,y)\vec{F}_1 + c_2(t)s_2(x,y)\vec{F}_2 , \qquad 7.11$$

where $c_1(t)=e^{-Kt}$ and $c_2(t)=(1-e^{-Kt})$. If the flow is to be cyclonic for t large, we must choose $\mu<0$ and $\omega<0$. The ratio of μ to ω determines the angle which the wind makes with respect to the $\vec{\theta}$ direction. Observed values of this angle range from 10° to 45°. The period for $\vec{F}_2$ is about $|2\pi/\omega|$ so to describe a developing cyclonic flow we should take $1/K>>|2\pi/\omega|$.

The functions $s_1(x,y)$ and $s_2(x,y)$ are introduced in equation 7.11 to produce a spatial scaling. If we choose $s_1(x,y)=C(1-e^{-|y|})/|y|$ the wind profile at t=0 is as indicated in Figure 7.8.

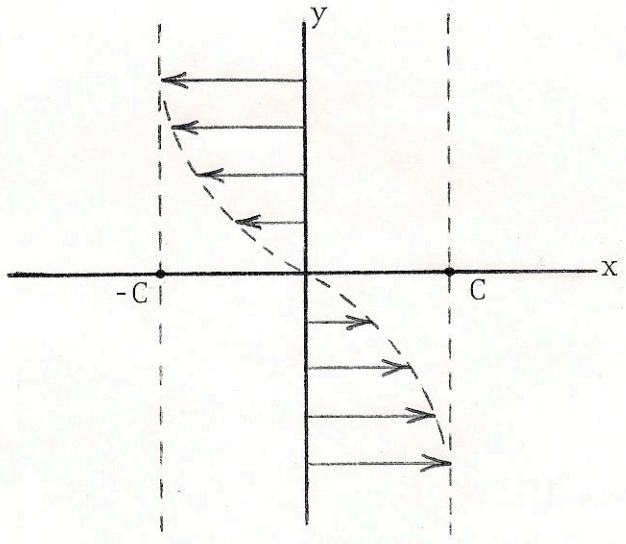


Figure 7.8: The wind velocity along any line parallel to the y axis at t=0.

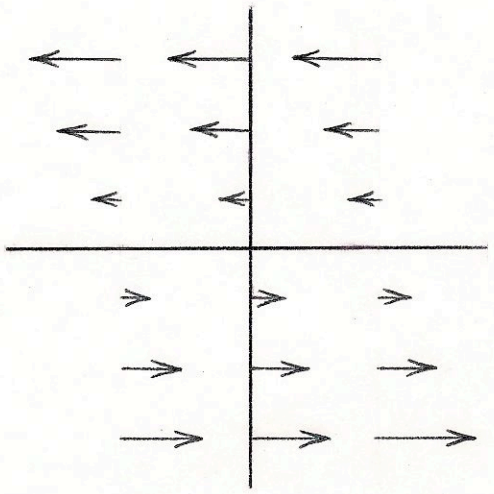
From meteorological observations it is known that for a purely cyclonic flow the wind velocity satisfies the following relation: $v\sqrt{r}$=constant, except near r=0. If we choose $s_2(x,y)=(1-e^{-r^2})/r^{3/2}$ we find that for Kt>>0 and r>>0, $v\sqrt{r}=\sqrt{\mu^2+\omega^2}$ while for Kt>>0 and r≈0, $v\approx\sqrt{r}$. Choosing $\mu=-.4$, $\omega=-1$, $1/K=10\pi$ and C=1 gives

$$\dot{x} = e^{-t/10\pi}(e^{-|y|}-1)y/|y| - (1-e^{-t/10\pi})(1-e^{-r^2})(.4x+y)/r^{3/2} \qquad 7.12$$

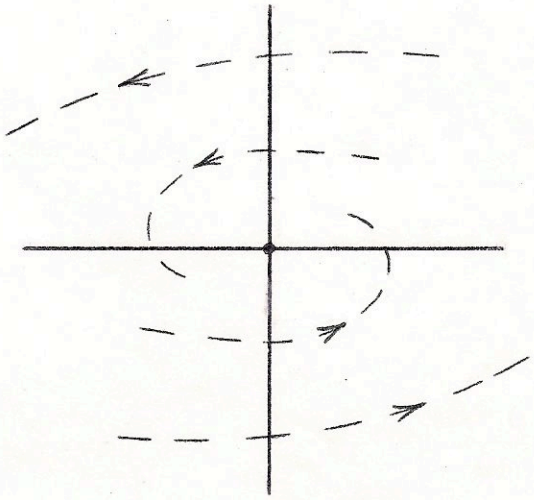
$$\dot{y} = (1-e^{-t/10\pi})(1-e^{-r^2})(x-.4y)/r^{3/2} \qquad 7.13$$

The flow given by equations 7.12 and 7.13 is shown at four stages of development in Figure 7.9. Notice that as $t\to+\infty$ the cyclone dissipates in that $v\to 0$ everywhere. To simulate a moving cyclone we need only change I to $(x-v_1t)+i(y-v_2t)$. The model given simulates the one first introduced by J. Bjerknes (see Petterssen [1], page 210 for an elementary discussion of Bjerknes' model).

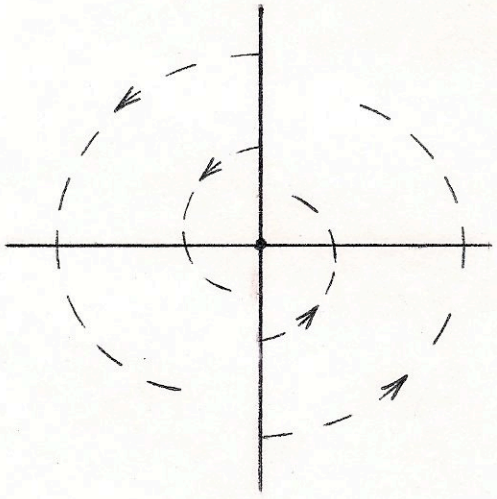
By making use of the theory developed in Chapter Five it is now possible to construct a relatively simple mathematical model for the wind field due to several lows and highs and to study how the various pressure centers interact. Of course the exact physical equations governing the motion of the molecules in the air can be



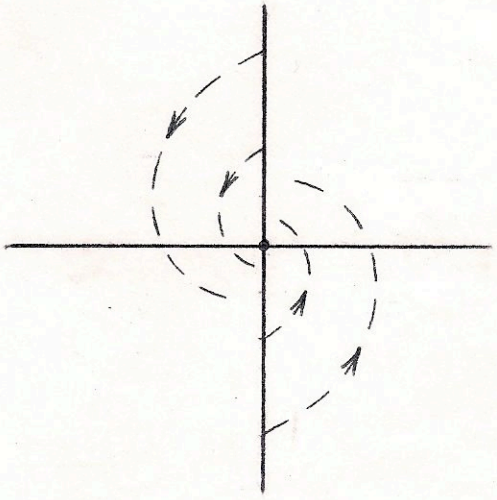
7.9.1: t=0



7.9.2: $t=5\pi=K/2$



7.9.3: $t=10\pi=K$



7.9.4: $t=30\pi=3K$

Figure 7.9: A mathematical model for extratropical cyclogenesis in the northern hemisphere.

written down but they are too complicated to solve and furthermore it is difficult to gather enough data to provide a well defined set of initial conditions. Models which only simulate the actual air flows depend on relatively few parameters and may prove useful for this reason.

Before we continue with a discussion of nonalgebraic autonomous flows we would like to point out that spatially scaled flows have been encountered in previous examples. See for instance the flows given in Figures 2.16.7, 2.16.10 and 7.3.1. In each of these cases the scaling factors were algebraic and led to lines of critical points. A natural way to produce algebraically scaled flows would be to relax the assumption that $P(x,y)$ has no multiple components; however algebraically scaled flows in themselves are not particularly interesting because of the simple way in which they are related to the associated unscaled flows. Only when scaled flows are combined in sum flows do they take on added interest.

As a second example of a scaled sum flow consider $\vec{F}=\vec{F}_1+s(r)\vec{F}_2$ where $\vec{F}_1=(y,-x)$, $\vec{F}_2=(x,y)$ and

$$s(r) = \frac{(1/\sqrt{2\pi}-r)}{r^3}\cos^2(\pi^2 r^2).$$

We have

$$\dot{x} = y + x\frac{(1/\sqrt{2\pi}-r)}{r^3}\cos^2(\pi^2 r^2) \qquad 7.14$$

$$\dot{y} = -x + y\frac{(1/\sqrt{2\pi}-r)}{r^3}\cos^2(\pi^2 r^2) \qquad 7.15$$

or

$$\dot{r} = \frac{(1/\sqrt{2\pi}-r)}{r^2}\cos^2(\pi^2 r^2) \ , \ \dot{\theta}=-1.$$

It is easy to see, by considering $\dot{r}$ and $\dot{\theta}$, that the flow given by equations 7.14 and 7.15 has a countable infinity of limit cycles located at $r=\sqrt{(2n-1)/2\pi}$ where n is a natural number. The first limit cycle is stable while the rest are semistable. As $r\to\infty$ the system goes to $\dot{x}=y$ and $\dot{y}=-x$, which corresponds to the equations of motion for a one dimension classical harmonic oscillator where all of the physical parameters have been set equal to one (see Exercise 13). Equations 7.14 and 7.15 simulate the significant features of the one dimensional quantum harmonic oscillator. See Figure 7.10.

An alternate way of producing nonalgebraic flows is by means of transformations of the plane into itself. For example consider the flow around the Joukowski airfoil (see Pennisi [1], page 385 for a discussion of this particular flow). The Joukowski airfoil is of interest

because of its similarity in shape to the cross-section of an airplane wing. See Figure 7.11.

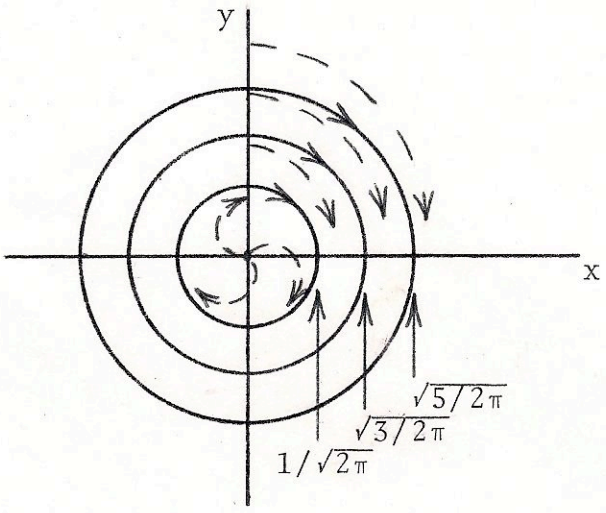


Figure 7.10: The flow given by equations 7.14 and 7.15.

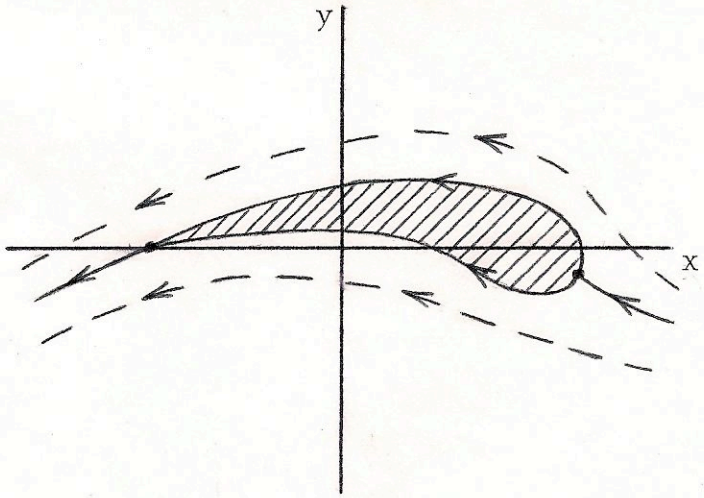


Figure 7.11: Flow around the Joukowski airfoil.

To arrive at the Joukowski airfoil consider a circle of radius ρ in the complex plane centered at z_0, passing through $z=-k$ and containing the point $z=k$ in its interior where k is real and $k>0$. The image of this circle under the transformation $w=z+k^2/z$ is the Joukowski airfoil. Finding the correct flow amounts to finding a flow in E^2 which satisfies the following set of conditions:

i) $x^2+y^2=\rho^2$ is a solution curve.

ii) $\vec{F}$ goes to a constant vector as r goes to infinity.

iii) $\int_C \vec{F}_n ds=0$ for every closed contour c whose interior lies outside of $x^2+y^2=\rho^2$.

iv) $\int_C \vec{F}_t ds=K$ for every closed contour c which encloses $x^2+y^2=\rho^2$.

The line integral which occurs in iii) was defined in Chapter Five as the flux. Condition iii) says that there should be no sources outside of $x^2+y^2=\rho^2$. The line integral in condition iv) defines the <u>circulation</u> of the flow. We will next show how to construct a flow $\vec{F}$ which satisfies conditions i) through iv).

We begin by constructing a flow, $\vec{F}_1$, which satisfies i), ii) and iii) and has circulation zero. We then construct a second flow, $\vec{F}_2$, which satisfies i), ii) and iv) for which $\vec{F}_2 \to \vec{0}$ as $r \to \infty$. The desired flow is $\vec{F}_1+\vec{F}_2$. Without loss of generality suppose that the flow at ∞ is

in the -x direction. If we let $\vec{F}_1$ be the class I flow generated by $L_1=x^2+y^2-\rho^2$ and $L_2=y$ then we cannot satisfy both conditions ii) and iii), even if we allow scaling.

Exercise 10: Verify the preceeding assertion.

Suppose that we now add the generator I=x+iy and scale the resulting class I flow by $1/r^4$. We then have for $\vec{F}_1$:

$$\dot{x} = 2\mathrm{Re}(Bi\bar{I})y(x^2+y^2-\rho^2)/r^4 + (A_1 2y^2+A_2(x^2+y^2-\rho^2))(x^2+y^2)/r^4$$

$$\dot{y} = -2\mathrm{Re}(B\bar{I})y(x^2+y^2-\rho^2)/r^4 - A_1 2xy(x^2+y^2)/r^4 .$$

On x=0, $\dot{x}\to 2\mathrm{Re}(B)+2A_1+A_2$ and $\dot{y}\to -2\mathrm{Im}(B)$ as $y\to+\infty$. On y=0, $\dot{x}\to A_2$ and $\dot{y}\to 0$ as $x\to+\infty$. If we choose $\mathrm{Im}(B)=0$ and $\mathrm{Re}(B)=-A_1$ condition ii) is satisfied. Condition iii) will be met provided $\vec{\nabla}\cdot\vec{F}_1=0$. We find that $\vec{\nabla}\cdot\vec{F}_1=2\rho^2\cos\theta(\mathrm{Re}(B)+A_2)/r^3$ so we must choose $\mathrm{Re}(B)=-A_2$. Finally $\vec{F}_1$ will be circulation free provided $\vec{\nabla}\times\vec{F}_1=\vec{0}$. A calculation yields $\vec{\nabla}\times\vec{F}_1=-\hat{z}2\rho^2\sin\theta(\mathrm{Re}(B)+A_2)/r^3$ so $\vec{\nabla}\times\vec{F}_1=\vec{0}$ as required.

We can arrive at $\vec{F}_2$ by scaling the class I flow generated by I=x+iy, where $\Lambda=-iK/2\pi$, by $1/r^2$. The desired flow, $\vec{F}$, is then given by

$$\dot{x}= A_2(1-\rho^2/r^2) + 2A_2\rho^2y^2/r^4 - Ky/(2\pi r^2) \qquad 7.16$$

$$\dot{y} = -2A_2\rho^2xy/r^4 + Kx/(2\pi r^2). \qquad 7.17$$

See Figure 7.12. Now translate and rotate $\vec{F}$ so that the

circle of radius ρ is centered at z_0 and passes through $z=-k$. The image of this new flow under the transformation $w=z+k^2/z$ is the flow shown in Figure 7.11. The conditions i) through iv) are still met by the transformed flow.

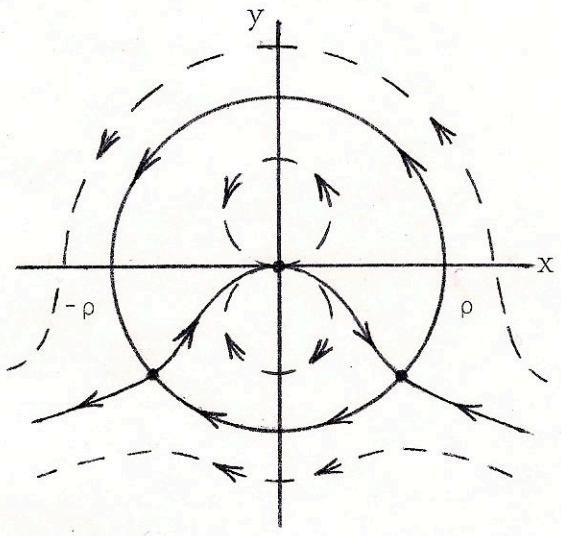


Figure 7.12: The flow given by equations 7.16 and 7.17.

It is interesting to observe that the flow given by equations 7.16 and 7.17 is uniquely determined by conditions i) through iv) and, A_2 and K. For a proof of this fact see Pennisi [1], page 374, Theorem 9.5.2. If we wanted to study flows close to $\vec{F}$ we could do so by relaxing conditions ii), iii) and iv) in a systematic way as follows. We could let $Im(B)\approx 0$ or $Re(B)\approx -A_1$ or

$Re(B) \tilde{} -A_2$. More complicated neighboring flows could be obtained by introducing extra generators or by considering extension flows.

There are many types of transformations which can be used to produce nonalgebraic flows from algebraic flows generated by the master equation, the Schwarz-Christoffel transformation being just one example. In actual applications, in such fields as hydrodynamics, electricity, aerodynamics and the conduction of heat, the preimage flow is usually a sum flow whose constituents are relatively simple scaled class I flows. The simplicity of the constituent flows is due to the imposition of side conditions dictated by physical laws.

Another way of producing nonalgebraic flows would be to use the master equation but to let the generators be nonalgebraic. A complete study of such flows would undoubtedly be difficult; however, since nonalgebraic curves can usually be approximated by algebraic curves we would expect many of the concepts and theorems presented here to generalize to nonalgebraic flows. We now give one simple example of a flow generated by nonalgebraic curves.

Let $P_1=y-K\cos(x/2)$ and $P_2=y+K\cos(x/2)$, where $K=2\sqrt{g/\ell}$. If $-\Lambda_1(\pi,0)=\Lambda_2(\pi,0)=K/2$ then the class I flow

generated by P_1 and P_2 is

$$\dot{x} = y \qquad 7.18$$

$$\dot{y} = -(g/\ell)\sin x \,. \qquad 7.19$$

Exercise 11: Show that equations 7.18 and 7.19 are correct as claimed.

Exercise 12: Show that equations 7.18 and 7.19 are also generated by I and $\overline{I}$ where $I=K\sin(x/2)+iy$ and $\Lambda(0,0)=i\sqrt{g/\ell}$.

The reader will perhaps recognize equations 7.18 and 7.19 as the equations of motion for a simple pendulum where x is the angular displacement, y is the angular velocity, g is the acceleration due to gravity and ℓ is the length of the support. The generators can be arrived at by solving the energy equation. It should be noted that the angular frequency, $\sqrt{g/\ell}$, which is an important physical parameter, is proportional to the local rate.

Exercise 13: Find a pair of generators and local rates which will produce the flow corresponding to a simple harmonic oscillator. Identify the physical significance of the local rates.

A final generalization of the master equation which

we will mention only briefly is to spaces of dimension higher than two. Because of its form the master equation can only be generalized to spaces of even dimension. Suppose that

$$P = \prod_{k=1}^{m} P_k(x_1,\cdots,x_n,y_1,\cdots,y_n),$$

where the P_k's are distinct irreducible algebraic functions of the 2n variables, x_1 through y_n, and n>1. Let the master equation in E^{2n} be given by

$$\dot{x}_i = \sum_{k=1}^{m} A_{ki} P_{ky_i} \prod_{j\neq k}^{m} P_j + A_{\infty i} \prod_{j=1}^{m} P_j \qquad 7.20$$

$$\dot{y}_i = -\sum_{k=1}^{m} A_{ki} P_{kx_i} \prod_{j\neq k}^{m} P_j - B_{\infty i} \prod_{j=1}^{m} P_j , \qquad 7.21$$

where i=1,$\cdots$,n.

Exercise 14: Show that each component of P is a solution to the system given by equations 7.20 and 7.21.

Since it is not possible to define tangent hyperplanes at multiple points of the hypersurfaces defined by the P_k's, the concept of local rates does not generalize past E^2. Global rates however remain meaningful.

Many examples of systems of the same general form as those given by equations 7.20 and 7.21 occur in classical mechanics since 7.20 and 7.21 reduce to

example it has been shown that the linear flows and the homogeneous flows of degree n are all generated by the master equation. The results which have been achieved have lead us to believe that the approach taken may prove quite fruitful, however much work remains to be done. What follows is a list of problems whose solutions may help to solve the central question posed above. It is obviously difficult to prejudge the value of a mathematics problem. Furthermore some of the problems may be improperly stated and the solutions to others may necessitate the introduction of new concepts. Nonetheless the problems seem interesting to us and it is our hope that they will be of interest to other mathematicians.

Several of the previously studied flows appeared to be class III flows. In particular the extension flow given by $\dot{x}=x(-1+x+1.1y)$, $\dot{y}=2y(1-2x-y)$, (see Figure 4.1); the sum flow given by $\dot{x}=-x$, $\dot{y}=y(2-3x-y)$, (see Figure 4.2); and the extension flow given by $\dot{x}=y(x+y+2)+(x^2+y^2-1)$, $\dot{y}=-x(x+y+2)$, (see Figure 7.5) were cited as possible class III flows. However one cannot be certain that they are class III flows since in other cases extension and sum flows have turned out to be equivalent to class I or II flows. In fact, in the linear case, all of the extension

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and sum flows were of class I or II. These observations lead us to the first problem.

Problem 1: Are there any algebraic flows which are not equivalent to those given by the master equation?

We observed that all quadratic flows can be represented as sum flows (Lemma 7.3). If the answer to Problem 1 turns out to be yes the following problem would warrant special consideration.

Problem 2: Is every algebraic flow equivalent to a sum of class I and/or class II flows?

Problem 2 is of some interest in itself since if the answer to it is yes we could then consider the class I and II flows to be the building blocks for all of the algebraic flows. If Problem 1 must be answered with a no, then of course the answer to Problem 2 is yes, in the trivial sense, and Problem 3 must be considered.

Problem 3: If a given algebraic flow is not of class III how can one find all sets of generating curves?

We should also consider the related problem.

Problem 4: Given that an algebraic flow has a set of generators is there any sense in which it has a simplest set of generators?

Examples of flows with more than one set of generators have already been encountered so Problem 4 seems to be a valid question. We conclude with six problems relating directly to class I and II flows.

Problem 5: Are there class I and class II flows which are equivalent?

Problem 6: Under what conditions is a generator a separatrix?

Problem 7: How can the presence of spontaneous separatrices be determined?

Problem 8: Is the set of flows generated by the master equation a connected set?

Problem 9: Is every flow generated by the master equation locally generated?

Problem 10: How should the algebraically scaled flows be classified?