CHAPTER 1

ALGEBRAICALLY GENERATED FLOWS

A. Classes of Flows in E^2

Suppose that we wish to find the simplest non-trivial class of first order systems of differential equations in the Euclidean plane, E^2, which have a given algebraic curve, $P(x,y)=0$, among their solutions. The curve can always be written in terms of its n irreducible components in the complex field as

$$P(x,y) = \prod_{i=1}^{n} P_i(x,y) \quad .$$

For the sake of simplicity let us assume that $P(x,y)$ has no multiple components, that each component has real points, and that complex components occur in conjugate pairs.

Systems of differential equations are classified in increasing order of complexity as linear, algebraic, analytic, differentiable or continuous. If we set

$$\dot{x} = P_y = \sum_{i=1}^{n} P_{iy} \prod_{j\neq i}^{n} P_j$$

$$\dot{y} = -P_x = -\sum_{i=1}^{n} P_{ix} \prod_{j\neq i}^{n} P_j \quad ,$$

then $P(x,y)=C$ is the general solution since $P_x\dot{x} + P_y\dot{y} = 0$. The system is linear if $n=2$ and P_1 and P_2 are linear, and algebraic otherwise. As a class of systems it is trivial, primarily because it contains only one member, but also because the general solution is known.

We may get a nontrivial class with the same degree of complexity by letting

$$\dot{x} = \sum_{i=1}^{n} A_i P_{iy} \prod_{j\neq i}^{n} P_j \qquad 1.1$$

$$\dot{y} = -\sum_{i=1}^{n} A_i P_{ix} \prod_{j\neq i}^{n} P_j \quad , \qquad 1.2$$

where the A_i's are nonzero constants. On P_i,

$\dot{x} = A_i P_{iy} \prod_{j\neq i}^{n} P_j$ and $\dot{y} = -A_i P_{ix} \prod_{j\neq i}^{n} P_j$, since P_i has real points, and $P_{ix}\dot{x} + P_{iy}\dot{y} = A_i \prod_{j\neq i}^{n} P_j (P_{ix}P_{iy} - P_{iy}P_{ix}) = 0$.

So $P_i(x,y)=0$ is a solution, and $P(x,y)=0$ is a solution.

There is no nontrivial class of less complexity for which $P(x,y)=0$ is a solution but there are an infinity of more complex classes. For example, consider

$$\dot{x} = \sum_{i=1}^{n} A_i P_{iy} \prod_{j \neq i}^{n} P_j + A_\infty P \qquad 1.3$$

$$\dot{y} = - \sum_{i=1}^{n} A_i P_{ix} \prod_{j \neq i}^{n} P_j - B_\infty P \qquad 1.4$$

where the A_i's are still nonzero constants and A_∞ and B_∞ are constant but not both zero. Let $D(P)$ be the degree of $P(x,y)$. The curves $P_i(x,y)=0$ are still solutions, but since

$$D(P_{iy}) \leq D(P_i)-1 \text{ and } D\left(\sum_{i=1}^{n} A_i P_{iy} \prod_{j \neq i}^{n} P_j\right) \leq D(P)-1 ,$$

the systems given by 1.3 , 1.4 are the next most complex after those given by 1.1, 1.2. Classes of higher complexity could be obtained by letting the A_i's be algebraic functions or by letting A_∞ and B_∞ be algebraic functions. We consider these systems in Chapter Four.

Let us define class I systems or flows as those given by equations 1.1, 1.2 or their limits, class II systems or flows as those given by equations 1.3, 1.4 or their limits, and class III systems or flows as algebraic flows not of class I or II. Limit flows will be defined and discussed fully in Chapter Three. Class III systems have not been given as detailed a definition as class I or II systems for two reasons. First, defining a more detailed classification scheme can only be justified after class I and II systems have been shown to have interesting and useful properties. Secondly, it may be possible to show that class III systems are actually topologically equivalent to systems of class I or II.

It is necessary to introduce class II systems because in the case of parallel components the A_i's of equations 1.1 and 1.2 lose most of their effect on the flow. To see this notice that when all of the P_{ix}'s are equal and all of the P_{iy}'s are equal, equations 1.1, 1.2 imply that $dy/dx=-P_{ix}/P_{iy}$, provided $P_{iy}\neq 0$. The general solution is then the parallel flow $P_i(x,y)+c$. Equations 1.3 and 1.4 lead to nonparallel flows in the same case.

B. The Master Equation

Call the system of equations given by 1.1, 1.2 or 1.3, 1.4 the master equation. Points at which $\dot{x}=\dot{y}=0$ are called critical points. We say that flows given by the master equation are generated by $P(x,y)$ and we call the irreducible components of $P(x,y)$ the generators.

Suppose that $P_i(0,0)=0$, then we can always write

$$P_i(x,y) = \sum_{k=m_i}^{n_i} P_{ik}(x,y) \quad ,$$

where P_{ik} is a homogeneous polynomial of degree k, $P_{im_i}\neq 0$. m_i is the multiplicity of P_i at (0,0). If $m_i=1$, (0,0) is a simple point of P_i, and P_{i1} is the tangent line to P_i at (0,0). We can always factor P_{im_i} into linear factors over the complex field, that is

$$P_{im_i}(x,y) = \prod_{k=1} L_{ik}(x,y)^{r_{ik}} ,$$

where $L_{ik}(x,y) = a_{ik}x + b_{ik}y$. The lines L_{ik} are called the _tangent lines_ to P_i at (0,0); r_{ik} is the multiplicity of the line L_{ik}. If P_i has more than one tangent line at (0,0), (0,0) is called a _multiple point_. Since we have assumed that P has no multiple components, the _multiplicity of P at (0,0)_ is $m = \sum_{i=1}^{n} m_i$. It should be noted that m is the smallest integer such that m=i+j and $\partial^m P/\partial x^i \partial y^j \neq 0$ at (0,0). The definitions just given may be extended to any point on P_i, say $p=(x_i,y_i)$, by a translation of the coordinate axes which takes (0,0) to (x_i,y_i). With this extension in mind we have the following theorem.

Theorem 1.1: Suppose $p \varepsilon P(x,y)$. Then p is a multiple point of P(x,y) if and only if it is a critical point of the master equation.

Proof: Suppose $p \varepsilon P_i$ only. Now on P_i, $\dot{x} = A_i P_{iy} \prod_{j \neq i}^{n} P_j$, $\dot{y} = -A_i P_{ix} \prod_{j \neq i}^{n} P_j$. If p is a multiple point $P_x(p)=P_y(p)=0$, and on P_i, $P_x = P_{ix} \prod_{j \neq i}^{n} P_j$, $P_y = P_{iy} \prod_{j \neq i}^{n} P_j$. Hence $\dot{x}(p)=\dot{y}(p)=0$. Conversely if p is a critical point, on P_i we have

$$\dot{x} = A_i P_{iy} \prod_{j\neq i}^{n} P_j = 0 \quad \text{and} \quad \dot{y} = -A_i P_{ix} \prod_{j\neq i}^{n} P_j = 0.$$

But $\prod_{j\neq i}^{n} P_j(p)\neq 0$ and $A_i\neq 0$ so $P_{ix}(p)=P_{iy}(p)=0$.

Therefore p is a multiple point. If p belongs to more than one component then P(x,y) has more than one tangent at p, that is, p is a multiple point. But we have $\dot{x}(p)=\dot{y}(p)=0$, so p is also a critical point.

Q.E.D.

More can be said concerning the number of critical points which the master equation may have with the help of the following two lemmas, both of which are corollaries of the well known theorem of Bezout (Fulton, p. 112).

Lemma 1.1: If two plane curves of degrees $D(P_i)$, $D(P_j)$ have more than $D(P_i)D(P_j)$ points in common then they have a common component.

Lemma 1.2: If P is a plane curve with n irreducible components then

$$\sum_{p_j} m(p_j)(m(p_j)-1) \leq (D(P)-1)(D(P)-2) + 2(n-1),$$

where $m(p_j)$ is the multiplicity of P at p_j.

Let us call the critical points which lie on P(x,y) natural critical points and those which do not, spontaneous critical points.

Theorem 1.2: The master equation has at most $D(P)(D(P)-3)/2 + n$, natural critical points (in E^2).

Proof: Each critical point is a multiple point by Theorem 1.1. Now suppose that the multiplicity of each multiple point is as small as possible, namely two. Now use Lemma 1.2.

Q.E.D.

Let $\dot{x}=X(x,y)$, $\dot{y}=Y(x,y)$, where X and Y are given by equations 1.1, 1.2 or 1.3, 1.4. Lemma 1.1 leads to:

Theorem 1.3: Class I flows have at most $D(X)D(Y)$ isolated critical points. Class II flows have at most $D(X)D(Y)$ critical points, all of which are isolated.

Proof: Consider a class I flow. Suppose X and Y have no common components, then X and Y have at most $D(X)D(Y)$ points in common, that is there are at most $D(X)D(Y)$ critical points. If X and Y have a common component, say F, consider X/F and Y/F. Then the number of isolated critical points is at most $D(X/F)D(Y/F)$, which is less than $D(X)D(Y)$. Every point on F is of course a critical point, hence the number of nonisolated critical points is infinite if F is a continuous curve in E^2.

Consider a class II flow. Suppose $A_\infty \neq 0$ and $F|X$, where F is a nonconstant polynomial. Then F contains a factor of P_i, for some i, since $F| \prod_{j=i}^{n} P_j$ and the P_j's are irreducible. But $P_{iy} \prod_{j\neq i}^{n} P_j$ does not contain a factor of P_i so $F \nmid P_{iy} \prod_{j\neq i}^{n} P_j$. Therefore $F \nmid X$. Hence X is irreducible or Y is irreducible or both. In any event X and Y have no common components so we can have at most D(X)D(Y) critical points, and they are isolated.

Q.E.D.

It should be noted that class I flows may have nonisolated critical points, for example when all of the P_i's are parallel, while class II flows cannot.

Next we make some observations concerning P_i.

Theorem 1.4: a) P_i is real if and only if P_{ik} is real.

b) If P_{im_i} is complex then P_i is complex.

c) If $P_j = \overline{P}_i$ then $L_{jk} = \overline{L}_{ik}$.

Proof: The P_{ik}'s are homogeneous polynomials of degree k so they are independent of each other. Hence a) and b) are true. The truth of c) is obvious from the definition of L_{ik}.

Q.E.D.

The converse of b) is false. For example let $P_1=x+iy^2$ so $P_{11}=x$, then P_1 is complex but P_{11} is real. The converse of c) is also false. Let $P_2=x+iy^3$, then $L_{21}=x=\overline{L}_{11}$ but $P_2\neq\overline{P}_1$.

Since X and Y are to be real we may have to place some restrictions on the coefficients A_i, A_∞ and B_∞.

<u>Theorem 1.5</u>: If X and Y, as given by the master equation, and P_i are all real then A_i, A_∞ and B_∞ are real. Furthermore if $A_j=\overline{A}_i$, when P_j is complex and $P_j=\overline{P}_i$, then X and Y are real.

Proof: Consider X as given by equation 1.3. Without loss of generality suppose that the pairs of conjugate components of P are P_1, P_2; P_3, P_4; $\cdots$; P_{2k-1}, P_{2k}, where $2k\leq n$. A simple calculation yields

$$X-\overline{X} = \sum_{i\ \mathrm{odd}}^{2k-1} ((A_i-\overline{A}_{i+1})P_{iy}\overline{P}_i+(A_{i+1}-\overline{A}_i)\overline{P}_{iy}P_i) \prod_{j\neq i,i+1}^{n} P_j$$

$$+ \sum_{i=2k+1}^{n} (A_i-\overline{A}_i)P_{iy} \prod_{j\neq i}^{n} P_j + (A_\infty-\overline{A}_\infty)P \ .$$

The last term in $X-\overline{X}$ is the one of largest degree so if X is real then $A_\infty=\overline{A}_\infty$. A similar calculation with $Y-\overline{Y}$ shows that $B_\infty=\overline{B}_\infty$. Now consider $X-\overline{X}$ on P_i where P_i is real. We have

$$(X-\overline{X})|_{P_i} = (A_i-\overline{A}_i)P_{iy} \prod_{j\neq i}^{n} P_j \ .$$

Since P has no multiple components, setting $(X-\overline{X})|_{P_i}=0$ implies that $A_i-\overline{A}_i=0$ or $P_{iy}|_{P_i}=0$. If the latter condition holds consider $(Y-\overline{Y})|_{P_i}$. $P_{iy}|_{P_i}$ and $P_{ix}|_{P_i}$ cannot both be identically zero so $A_i-\overline{A}_i=0$, that is, A_i is real. Finally, $A_{i+1}=\overline{A}_i$ for $i\leq 2k-1$ and odd, implies that X and Y are real

Q.E.D.

It will be shown in Chapter Five that if X and Y are real then $A_j=\overline{A}_i$ when P_j is complex and $P_j=\overline{P}_i$, provided P_j is linear.

C. Global and Local Rates

In order to gain a better understanding of the role played by the A_i's let us consider the rate of flow on tangents of the generators. Define Θ_{ik} by

$$\sin\Theta_{ik} = -\tilde{a}_{ik}$$
$$\cos\Theta_{ik} = \tilde{b}_{ik} \quad ,$$

where $\tilde{a}_{ik} = a_{ik}/\sqrt{|a_{ik}|^2 + |b_{ik}|^2}$ and $\tilde{b}_{ik} = b_{ik}/\sqrt{|a_{ik}|^2 + |b_{ik}|^2}$. In polar coordinates L_{ik} becomes $L_{ik} = r\sqrt{|a_{ik}|^2 + |b_{ik}|^2}\,\sin(\Theta-\Theta_{ik})$.

To be on L_{ik} means that $\Theta=\Theta_{ik}$ or $x=r\tilde{b}_{ik}$, $y=-r\tilde{a}_{ik}$. Note that Θ_{ik} may be complex so the point $(x,y)\varepsilon L_{ik}$ may not be in E^2.

If f(r) is a polynomial in r let $f(r)^*$ be the coefficient of the term of f(r) of lowest degree and designate the degree of the lowest degree term of f by q*. Call $\Lambda_{ik}(0,0)$ the local rate on P_i at (0,0) in the Θ_{ik} direction, and define $\Lambda_{ik}(0,0)$ by

$$\Lambda_{ik}(0,0) = A_i \prod_{j\neq i}^{n} P_j(r\tilde{b}_{ik}, -r\tilde{a}_{ik})^* \times$$

$$\times \left(|P_{ix}(r\tilde{b}_{ik}, -r\tilde{a}_{ik})|^2 + |P_{iy}(r\tilde{b}_{ik}, -r\tilde{a}_{ik})|^2\right)^{*1/2}.$$

To extend the definition of Λ_{ik} to an arbitrary point $p=(x_i,y_i)\varepsilon P_i$, we may translate the origin to the point p by letting $x=\underline{x}+x_i$, $y=\underline{y}+y_i$, and then apply the equation in the $\underline{x},\underline{y}$ coordinate system. We find

$$\Lambda_{ik}(x_i,y_i) = A_i \prod_{j\neq i}^{n} P_j(r\tilde{b}_{ik}+x_i,-r\tilde{a}_{ik}+y_i)^* \times$$

$$\times \left(|P_{ix}(r\tilde{b}_{ik}+x_i,-r\tilde{a}_{ik}+y_i)|^2\right.$$

$$\left.+ |P_{iy}(r\tilde{b}_{ik}+x_i,-r\tilde{a}_{ik}+y_i)|^2\right)^{*1/2} , \quad 1.5$$

where the kth tangent line to P_i at (x_i,y_i) is

$L_{ik} = a_{ik}(x-x_i)+b_{ik}(y-y_i)$.

The definition of a local rate on a solution curve may be extended to any analytic flow as follows.

Suppose P(x,y) is a solution curve which goes through the origin and let $L_k = r\sqrt{|a_k|^2+|b_k|^2}\sin(\Theta-\Theta_k)$ be the kth tangent to P(x,y) at (0,0). Let

$$\Lambda_{pk}(0,0) = (X(r\tilde{b}_k,-r\tilde{a}_k)^*/P_y(r\tilde{b}_k,-r\tilde{a}_k)^*) \times$$

$$\times (|P_x(r\tilde{b}_k,-r\tilde{a}_k)|^2 + |P_y(r\tilde{b}_k,-r\tilde{a}_k)|^2)^{*1/2}$$

This formula reduces to equation 1.5 when the flow is generated by the master equation and P is the ith generator, provided no other generator is tangent to L_k at (0,0). In the general case, since X and P are analytic and the degree of the numerator is larger than the degree of the denominator, the formula is well defined. However, when we do not know P(x,y), Λ_{pk} cannot be calculated. It is sometimes useful to notice that

$$\Lambda_{pk}(0,0) = \begin{cases} X(r\tilde{b}_k,-r\tilde{a}_k)^*/\cos\Theta_k & \Theta_k \neq \pm\pi/2 \\ Y(r\tilde{b}_k,-r\tilde{a}_k)^*/\sin\Theta_k & \Theta_k \neq 0 \end{cases}$$

We also have

$$X(r\tilde{b}_k,-r\tilde{a}_k)^* = X(x,-a_kx/b_k)^*\tilde{b}_k{}^{q^*}$$

if $b_k \neq 0$ and

$$Y(r\tilde{b}_k,-r\tilde{a}_k)^* = Y(-b_ky/a_k,y)^*(-\tilde{a}_k)^{q^*}$$

if $a_k \neq 0$. So if <u>$q^*=1$</u>

$$\Lambda_{pk}(0,0) = \begin{cases} X(x,-a_k x/b_k)^* & b_k \neq 0 \\ Y(-b_k y/a_k, y)^* & a_k \neq 0 \end{cases}$$

If L_k is the kth tangent to several generators at (0,0) then

$$\Lambda_{pk}(0,0) = \sum_i \Lambda_{ik}(0,0) \quad ,$$

where P is any one of the generators and the sum is taken over all of the generators tangent to L_k at (0,0).

The motivation for the definition of Λ_{ik} can be seen by considering the next theorem.

<u>Theorem 1.6</u>: If p is a simple point of P then

$$\Lambda_i(p) = A_i \prod_{j \neq i}^{n} P_j(p)\sqrt{a_i^2 + b_i^2} = \sqrt{\dot{x}^2(p) + \dot{y}^2(p)} \quad .$$

Proof: Since p is a simple point of P it can belong to just one component, say P_i, and P_i has a single tangent at p. $\Lambda_{ik}(p)$ can be written as $\Lambda_i(p)$. Translate the origin to p. Now near p, $P_{i\underline{x}} = a_i$ and $P_{i\underline{y}} = b_i$, where a_i and b_i are both real, so

$$(|P_{i\underline{x}}(\underline{r}\tilde{b}_i, -\underline{r}\tilde{a}_i)|^2 + |P_{i\underline{y}}(\underline{r}\tilde{b}_i, -\underline{r}\tilde{a}_i)|^2)^{*1/2}$$

$$= \sqrt{a_i^2 + b_i^2} \quad .$$

Furthermore $P_j(\underline{r}\tilde{b}_i, -\underline{r}\tilde{a}_i)^* = P_j(p)$ so

$$\Lambda_i(p) = A_i \prod_{j\neq i}^{n} P_j(p)\sqrt{a_i^2+b_i^2}. \quad \text{On } P_i$$

$$\dot{x} = A_i P_{iy} \prod_{j\neq i}^{n} P_j$$

$$\dot{y} = -A_i P_{ix} \prod_{j\neq i}^{n} P_j$$

$$\text{so } \dot{x}^2(p) + \dot{y}^2(p) = (A_i \prod_{j\neq i}^{n} P_j(p))^2(b_i^2+a_i^2) .$$

Q.E.D.

Theorem 1.6 has an interesting corollary which can be proven with the help of Theorem 1.2. From Theorem 1.2 we see that the multiple points on the generators are finite in number, and hence isolated.

Corollary 1.6: a) $\Lambda_i(p)$ is a continuous function of arc length on P_i at simple points.

b) The limit of $\Lambda_i(p)$ on P_i as p goes to a multiple point is always zero.

Proof: $\Lambda_i(p) = \sqrt{\dot{x}^2(p)+\dot{y}^2(p)}$ if p is a simple point of P_i. X and Y are algebraic, hence continuous in E^2. Therefore $\Lambda_i(p)$ is continuous on P_i at simple points. By Theorem 1.1 the multiple points of P are also critical points and $\dot{x}=\dot{y}=0$ at critical points so $\Lambda_i(p)$ goes to zero near critical points. Q.E.D.

From the definition of $\Lambda_{ik}(p)$ we see that Λ is multivalued at multiple points. Λ_{ik} is always proportional to A_i and $A_i \neq 0$ so $\Lambda_{ik} \neq 0$ unless $P_j = L_{ik}$ for some $j \neq i$. That is, Λ is usually discontinuous at multiple points. We shall see in later examples that Λ_{ik} may even be complex when A_i is real. Since A_i is associated with P_i and P_i generally consists of more than one point, we call A_i the <u>global rate on P_i</u>. Once the A_i's are fixed the Λ_{ik}'s are completely determined so the A_i's are in a sense the primary dynamic parameters of the flow. However we see from equation 1.5 that we may always think of replacing A_i by the appropriate constant times $\Lambda_{ik}(p)$ for any $p \varepsilon P_i$. Notice that once Λ_i is fixed at one point of P_i it is determined at every point of P_i. The set of Λ_i's then constitute an equivalent set of dynamic parameters of the flow. A_∞ and B_∞ are also global parameters and we shall investigate the role they play in the next section.

The parameter Λ_i is a generalization of the concept of an eigenvalue. We shall see in the next chapter that if the system is linear, with one critical point, then P has exactly two components, corresponding to two eigenvectors, and that there are two values for Λ at the critical point. These two values are equal to the two eigenvalues. In general it is the values of Λ at multiple points which play a significant role in determining the character of the flow. The value of Λ_i at simple points of P_i becomes important only when P_i has no multiple points. Exact formulas for Λ shall be given for special cases of interest in the following chapters.

D. Algebraic Flows in E^{2+}

To gain a better understanding of algebraic flows globally, Poincare ([1], p. 3) introduced a method by which the extended plane, E^{2+}, may be compactified. Geometrically the method is quite simple. Consider the unit sphere centered at the origin in E^3. Place the x-y plane on top of the sphere so that the origin is at u=v=0, z=1. Choose the axes so that the x axis is parallel to the u axis and the y axis is parallel to the v axis. Project each point in the x-y plane onto the sphere through (0,0,0). Call the upper hemisphere together with the equator the compactified plane and designate it by CE^{2+}. The flow in CE^{2+}, called the compactified flow, is simply the image under the Poincare projection of the flow in E^{2+}. We see that the flow at infinity projects onto the equator in the z=0 plane. If we wish to study the flow in E^2

it is convenient to project the compactified flow onto the unit disk in the u-v plane. See Figure 1.1.

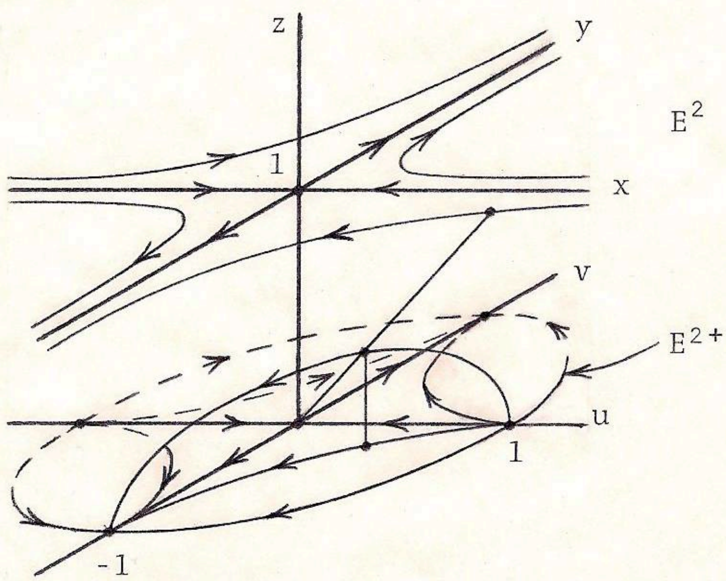


Figure 1.1: The compactification of E^{2+} and the flow in the compactified plane.

Algebraically the compactification is defined by

$$x=u/z,\ y=v/z,\ u^2+v^2+z^2=1.$$

Consider a straight line in the x-y plane given by $y=c$. In the u-v plane the line becomes $v=c\sqrt{1-u^2-v^2}$ or $u^2+v^2(1+c^2)/c^2=1$. If $c>0$ and $z>0$ then $v>0$ so the image curve is the upper half of an ellipse. See

Figure 1.2.

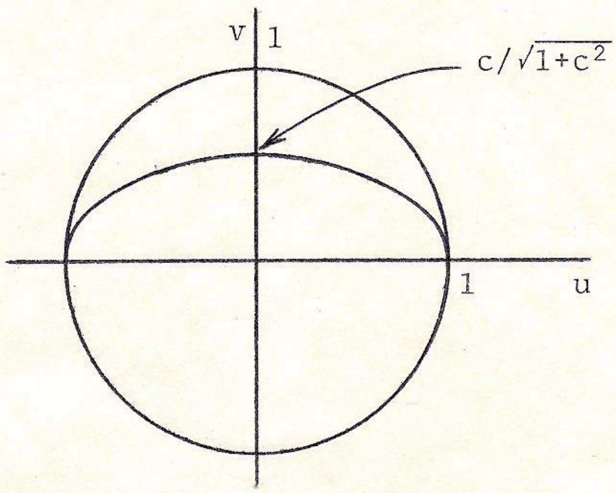


Figure 1.2: Image of a straight line in the unit disk.

If c=0 the corresponding curve is v=0. We see that the behavior of the image curve at points on $u^2+v^2=1$ depends on the location of the preimage curve in the x-y plane. This means that the flow at infinity cannot be easily studied in the unit disk.

If $P_i(x,y)$ is a curve in the x-y plane let us define the <u>corresponding compactified curve in the u-v plane</u>, $P_i^c(u,v)$, by

$$P_i^c(u,v) = z^{n_i} P_i(u/z, v/z) \quad ,$$

where $z=\sqrt{1-u^2-v^2} \geq 0$. We may also write

$$P_i^c(u,v) = \sum_{k=0}^{n_i-m_i} z^k P_{in_i-k}(u,v) \quad .$$

This definition is only valid when $P_i(x,y)$ is an algebraic function. $P_i^c(u,v)$ is $P_i(u,v)$ made homogeneous of degree n_i with z. As in the case of the straight line it may be necessary to square the equation in order to make the compactified curve an algebraic function of u and v. Let us call u^2+v^2-1, or z=0, the <u>line at infinity</u>, and designate it by $P_\infty(u,v)$.

The points on P_∞ which are also on $P_i^c(u,v)$ will be called <u>multiple points at infinity of $P_i(x,y)$</u>. These points are contained in the solution set of

$$P_{in_i}(u,v) = 0 \ , \ (u,v) \in P_\infty \ . \qquad 1.6$$

Notice that $P_{in_i}(-u,-v) = (-1)^{n_i} P_{in_i}(u,v)$ so the solutions of 1.6 correspond to symmetric pairs of points on P_∞. All of these points need not be connected with $P_i^c(u,v)$, z>0. For example if $P_i=y-x^2$ we have $n_i=2$ so $P_{in_i}(u,v)=-u^2$. Of the two multiple points (0,1) amd (0,-1), only the first is connected with $P_i^c(u,v)$, z>0. Both points however are <u>on</u> $P_i^c(u,v)$ because the definition includes z=0. See Figure 1.3.

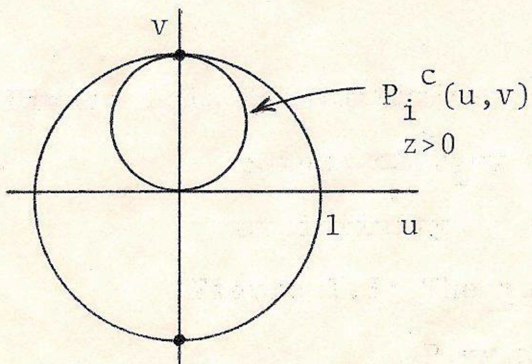


Figure 1.3: The multiple points of $P_i=y-x^2$ at infinity.

To find the flow at infinity notice that $x=u/z$ and $y=v/z$ imply

$$dx = (zdu-udz)/z^2$$

$$dy = (zdv-vdz)/z^2$$

Now $Xdy-Ydx=0$ becomes

$$-zY(u/z,v/z)du + zX(u/z,v/z)dv + (uY(u/z,v/z)-vX(u/z,v/z))dz = 0 \qquad 1.7$$

Equation 1.7 is not well defined at $z=0$ so let $N=\max(D(X),D(Y))$ and multiply through by z^N. We also have

$$u^2 + v^2 + z^2 = 1$$

so

$$udu + vdv + zdz = 0 \qquad 1.8$$

Letting

$$A(u,v,z) = -z^{N+1}Y(u/z,v/z)$$

$$B(u,v,z) = z^{N+1}X(u/z,v/z)$$

$$C(u,v,z) = z^N(uY(u/z,v/z)-vX(u/z,v/z))$$

we get, from 1.7 and 1.8,

$$\frac{du}{Bz-Cv} = \frac{dv}{Cu-Az} = \frac{dz}{Av-Bu} ,$$

or

$$du/d\tau = Bz-Cv$$

$$dv/d\tau = Cu-Az$$

$$dz/d\tau = Av-Bu .$$

For $z \geq 0$ this system gives the flow in CE^{2+}. Notice that it is homogeneous even when the original flow in the plane is not.

If r,Θ are polar coordinates in the $z=0$ plane, consider

$$r\, dr/d\tau = u\, du/d\tau + v\, dv/d\tau .$$

When $r=1$ we get

$$dr/d\tau = z(Bu-Av) = 0 ,$$

that is, P_∞ is always a solution to the system. To find the flow on P_∞ notice that

$$d\Theta/d\tau = u\, dv/d\tau - v\, du/d\tau = C(u,v,0) ,$$

when $z=0$. A point on P_∞ is called a critical point at infinity if

$$C(u,v,0) = 0 \text{ when } (u,v)\varepsilon P_\infty .$$

Critical points at infinity also occur in diametrically opposed pairs.

To see the relationship between multiple and critical points at infinity for flows generated by the master equation consider the next lemma.

Lemma 1.3: For a flow generated by the master equation

$$C(u,v,0) = \begin{cases} -\left(\sum_{i=1}^{n} A_i n_i\right) \prod_{j=1}^{n} P_{jn_j}(u,v) & \text{class I} \\ \\ -(A_\infty v + B_\infty u) \prod_{j=1}^{n} P_{jn_j}(u,v) & \text{class II} \end{cases}$$

Proof: $$xY-yX = -\sum_{i=1}^{n} A_i(xP_{ix}+yP_{iy}) \prod_{j\neq i}^{n} P_j - (A_\infty y+B_\infty x) \prod_{j=1}^{n} P_j$$

and $$xP_{ix}+yP_{iy} = \sum_{k=m_i}^{n_i} (xP_{ikx}+yP_{iky})$$

but $$xP_{ikx}+yP_{iky} = \sum_{j+\ell=k} a_{j\ell}(j+\ell)x^j y^\ell = kP_{ik} .$$

So
$$z^N(uY(u/z,v/z)-vX(u/z,v/z)) =$$

$$-z^{N+1} \sum_{i=1}^{n} A_i\left(\sum_{k=m_i}^{n_i} kP_{ik}(u/z,v/z)\right) \prod_{j\neq i}^{n} P_j(u/z,v/z)$$

$$-z^N(A_\infty v+B_\infty u) \prod_{j=1}^{n} P_j(u/z,v/z) = C(u,v,z).$$

Now set $z=0$ to get the desired result.

Q.E.D.

As a consequence of this lemma we may make several interesting observations.

Theorem 1.7: For a class I flow if $\sum_{i=1}^{n} A_i n_i \neq 0$ then:

a) There are a finite number of critical points at infinity; b) Each critical point at infinity is a multiple point, and conversely. If $\sum_{i=1}^{n} A_i n_i = 0$, then every point at infinity is a critical point. For a class II flow: a) There are only a finite number of critical points at infinity; b) Every critical point at infinity is a multiple point except for the two critical points given by $A_\infty v+B_\infty u=0$, $(u,v)\varepsilon P_\infty$.

Proof: Use Lemma 1.3. Q.E.D.

Call the critical points given by $A_\infty v+B_\infty u=0$, $(u,v)\varepsilon P_\infty$ <u>class II critical points</u>. We see from Theorem 1.7 that all of the isolated critical points at infinity should be considered to be natural critical points.

To study the flow at infinity it is necessary to project the flow from CE^{2+} onto a plane tangent to the sphere at the equator. Without loss of generality consider the tangent plane at u=1, v=0. This plane is called the <u>projective plane</u>. Let the $+\eta$ axis be in the same direction as the +v axis and the $+\xi$ axis be in the same direction as the +z axis. If (u,v,z) is a point on the unit sphere its projection onto the η-ξ plane is given by

$$\eta = v/u \ , \ \xi = z/u \ .$$

Using the equation Au+Bv+Cz=0 we have

$$d\eta/d\tau = C(u,v,z)/u^2 \ , \ d\xi/d\tau = -B(u,v,z)/u^2 . \qquad 1.9$$

If we now divide both equations by u^{N-1} and replace τu^{N-1} by σ we have the flow in the η-ξ plane. Let

$$X^h(\eta,\xi) = (z^N X(u/z,v/z))(1,\eta,\xi) \qquad 1.10$$

$$Y^h(\eta,\xi) = (z^N Y(u/z,v/z))(1,\eta,\xi) \qquad 1.11$$

Lemma 1.4: $d\eta/d\sigma = Y^h(\eta,\xi) - \eta X^h(\eta,\xi)$

$$d\xi/d\sigma = -\xi X^h(\eta,\xi)$$

Proof: Use equations 1.9, 1.10 and 1.11. Q.E.D.

If we wish to study the flow at some point (u^*,v^*) on the line at infinity other than (1,0) we may do so by considering the rotated flow. Let $\Theta = \tan^{-1}(v^*/u^*)$. The rotated flow is given by

$$\dot{x} = X_R(x,y) = X(\underline{x},\underline{y})\cos\Theta + Y(\underline{x},\underline{y})\sin\Theta$$

$$\dot{y} = Y_R(x,y) = -X(\underline{x},\underline{y})\sin\Theta + Y(\underline{x},\underline{y})\cos\Theta ,$$

where
$$\underline{x} = x\cos\Theta - y\sin\Theta$$

$$\underline{y} = x\sin\Theta + y\cos\Theta$$

The vector field (X_R,Y_R) is just the vector field (X,Y) rotated by $-\Theta$ degrees. We may now set

$$(d\eta/d\sigma)|_{(u^*,v^*)} = Y_R{}^h(\eta,\xi) - \eta X_R{}^h(\eta,\xi)$$

$$(d\xi/d\sigma)|_{(u^*,v^*)} = -\xi X_R{}^h(\eta,\xi) .$$

Let us now define the transformation T_Θ by

$$T_\Theta(X,Y) = ((d\eta/d\sigma)|_{(u^*,v^*)} , (d\xi/d\sigma)|_{(u^*,v^*)}) .$$

Since the rotated flow has the same topological characteristics as the original, we could make the equations for class I and class II flows look more similar in form by letting $B_\infty = A_\infty$, without losing any generality. This convention would simply place the class II critical points at $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}) \varepsilon P_\infty$. It should also be noted that the rotated flow has the same form as the original if the original is generated by the master equation. Let the <u>rotated curve</u>, P_{iR}, be given by

$$P_{iR}(x,y) = P_i(\underline{x},\underline{y}) \ . \qquad 1.12$$

Then

$$\dot{x} = X_R(x,y) = \sum_{i=1}^{n} A_i P_{iRy}(x,y) \prod_{j \neq i}^{n} P_{jR}(x,y) + A_{\infty R} \prod_{j=1}^{n} P_{jR}(x,y)$$

$$\dot{y} = Y_R(x,y) = -\sum_{i=1}^{n} A_i P_{iRx}(x,y) \prod_{j \neq i}^{n} P_{jR}(x,y) - B_{\infty R} \prod_{j=1}^{n} P_{jR}(x,y),$$

where
$$A_{\infty R} = A_\infty \cos\Theta - B_\infty \sin\Theta$$
$$B_{\infty R} = A_\infty \sin\Theta + B_\infty \cos\Theta \ .$$

Suppose we have a flow in the x-y plane given by $\dot{x}=X(x,y)$, $\dot{y}=Y(x,y)$. Consider the <u>translated flow</u> given by $\dot{x}=X_T(x,y)=X(x-a,y-b)$, $\dot{y}=Y_T(x,y)=Y(x-a,y-b)$. Since $D(X_T)=D(X)$ and $D(Y_T)=D(Y)$, $N_T=N$. Now $C(u,v,0)$ is equal to the N+1 degree terms of $uY-vX$ and $C_T(u,v,0)$ is equal to the N+1 degree terms of uY_T-vX_T, hence

$$C(u,v,0) \equiv C_T(u,v,0).$$

The two flows have exactly the same critical points at infintiy. The translation of the origin to (a,b), which takes the first flow into the second, induces a homeomorphism between the corresponding flows on the Poincare sphere and hence induces a homeomorphism between the flows in the projective plane. We have proved the following theorem.

Theorem 1.8: The flow at infinity (in the projective plane) is topologically independent of the position of the origin in the x-y plane.

The flow near points opposite one another on the line at infinity posseses a simple symmetry due to the nature of the Poincare projection.

Theorem 1.9: If the flow at $(u,v)\varepsilon P_\infty$ is given by

$$d\eta/d\sigma = R(\eta,\xi), \quad d\xi/d\sigma = T(\eta,\xi)$$

then the flow at (-u,-v) is given by

$$d\eta/d\sigma=(-1)^{N+1}R(\eta,-\xi), \quad d\xi/d\sigma=(-1)^{N}T(\eta,-\xi).$$

Proof: Without loss of generality let (u,v)=(1,0). To find the flow at (-1,0) rotate the original flow by -180°. The rotated flow is given by $\dot{x}=X_R(x,y)=-X(-x,-y)$, $\dot{y}=Y_R(x,y)=-Y(-x,-y)$. Now

$$X_R^h(\eta,\xi) = (Z^N X_R(u/z,v/z))(1,\eta,\xi)$$

$$= (-1)^{N+1}((-z)^N X(u/-z,v/-z))(1,\eta,\xi)$$

$$= (-1)^{N+1}(z^N X(u/z,v/z))(1,\eta,\xi)$$

$$= (-1)^{N+1}X^h(\eta,-\xi), \text{ and}$$

$Y_R^h(\eta,\xi) = (-1)^{N+1}Y^h(\eta,-\xi)$. Using Lemma 1.4 we have

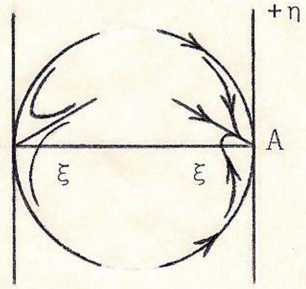
$$d\eta/d\sigma = (-1)^{N+1}(Y^h(\eta,-\xi)-\eta X^h(\eta,-\xi)) = (-1)^{N+1}R(\eta,-\xi)$$

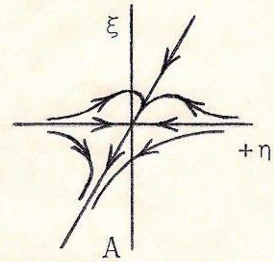
$$d\xi/d\sigma = -(-1)^{N+1}(-(-\xi)X^h(\eta,-\xi)) = -(-1)^{N+1}T(\eta,-\xi) \quad \text{or}$$

$$d\eta/d\sigma = (-1)^{N+1}R(\eta,-\xi) \ , \ d\xi/d\sigma = (-1)^N T(\eta,-\xi) \text{ as desired.}$$

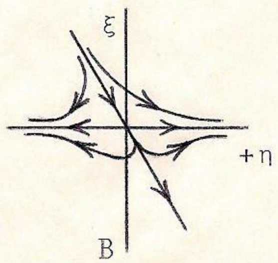
Q.E.D.

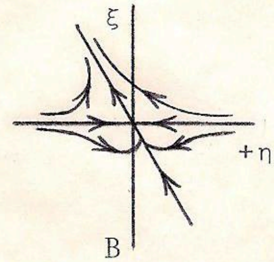
Notice that the second set of equations in Theorem 1.9 can be obtained from the first, if N is odd, by replacing ξ by $-\xi$ throughout; and if N is even, by replacing ξ by $-\xi$ and t by -t. N odd corresponds to a reflection about the η axis and N even corresponds to a reflection about the η axis plus a time reversal. See Figure 1.4.





$(R(\eta,\xi),\ T(\eta,\xi))$





$(-R(\eta,-\xi),\ T(\eta,-\xi))$, N even $\quad$ $(R(\eta,-\xi),\ -T(\eta,-\xi))$, N odd

Figure 1.4: The flow at symmetric points on P_∞.

Let
$$P_i^h(\eta,\xi) = (z^{n_i}P_i(u/z,v/z))(1,\eta,\xi)$$
$$= \sum_{k=0}^{n_i-m_i} \xi^k P_{in_i-k}(1,\eta). \qquad 1.13$$

<u>Theorem 1.10</u>: For a flow given by the master equation

$$d\eta/d\sigma\big|_{(1,0)} = -[(\xi \sum_{i=1}^{n} A_i n_i)+\eta A_\infty+B_\infty] \prod_{j=1}^{n} P_j^h(\eta,\xi)$$
$$+\xi^2 \sum_{i=1}^{n} A_i P_{i\xi}^h(\eta,\xi) \prod_{j\neq i}^{n} P_j^h(\eta,\xi)$$

$$d\xi/d\sigma\big|_{(1,0)} = -\xi[\xi \sum_{i=1}^{n} A_i P_{i\eta}^h(\eta,\xi) \prod_{j\neq i}^{n} P_j^h(\eta,\xi)+A_\infty \prod_{j=1}^{n} P_j^h(\eta,\xi)].$$

Proof: Suppose $A_\infty^2+B_\infty^2>0$, then $N=\sum_{i=1}^{n} n_i$. $Y^h(\eta,\xi)-\eta X^h(\eta,\xi)$ is $xY(x,y)-yX(x,y)$ made homogeneous of degree N+1 with z, evaluated at x=1, y=η, z=ξ.

$$xY-yX = -\sum_{i=1}^{n} A_i(xP_{ix}+yP_{iy})\prod_{j\neq i}^{n} P_j(x,y)$$

$$-(yA_\infty+xB_\infty)\prod_{j=1}^{n} P_j(x,y)$$

so

$$Y^h(\eta,\xi) - \eta X^h(\eta,\xi) =$$

$$-[z\sum_{i=1}^{n} A_i(xz^{n_i-1}P_{ix}(x/z,y/z)$$

$$+ yz^{n_i-1}P_{iy}(x/z,y/z))\prod_{j\neq i}^{n} z^{n_j}P_j(x/z,y/z)$$

$$+ (yA_\infty+xB_\infty)\prod_{j=1}^{n} z^{n_j}P_j(x/z,y/z)](1,\eta,\xi).$$

Using the fact that

$$xP_{ijx}(x,y) + yP_{ijy}(x,y) = jP_{ij}(x,y)$$

We may write

$$(xz^{n_i-1}P_{ix}(x/z,y/z)+yz^{n_i-1}P_{iy}(x/z,y/z))(1,\eta,\xi)$$

$$= (\sum_{k=0}^{n_i-m_i} z^k(xP_{in_i-k}(x,y)+yP_{in_i-k}(x,y)))(1,\eta,\xi)$$

$$= (\sum_{k=0}^{n_i-m_i} (n_i-k)z^kP_{in_i-k}(x,y))(1,\eta,\xi)$$

$$= n_i(z^{n_i}P_i(x/z,y/z))(1,\eta,\xi)$$

$$- (z\sum_{k=0}^{n_i-m_i} kz^{k-1}P_{in_i-k}(x,y))(1,\eta,\xi)$$

$$= n_iP_i^h(\eta,\xi) - \xi P_{i\xi}^h(\eta,\xi) .$$

Hence $d\eta/d\sigma|_{(1,0)}$ is as claimed.

$X^h(\eta,\xi)$ is $X(x,y)$ made homogeneous of degree N with z, evaluated at $x=1$, $y=\eta$, $z=\xi$.

$$X^h(\eta,\xi) = [z\sum_{i=1}^{n} A_i z^{n_i-1}P_{iy}(x/z,y/z)\prod_{j\neq i}^{n} z^{n_j}P_j(x/z,y/z)$$

$$+ A_\infty\prod_{j=1}^{n} z^{n_j}(x/z,y/z)](1,\eta,\xi) .$$

Now

$$(z^{n_i-1}P_{iy}(x/z,y/z))(1,\eta,\xi)$$

$$= (\sum_{k=0}^{n_i-m_i} z^k(P_{in_i-k})_y(x,y))(1,\eta,\xi)$$

$$= \sum_{k=0}^{n_i-m_i} \xi^k(P_{in_i-k})_\eta(1,\eta) = P_{i\eta}^h(\eta,\xi).$$

Hence $d\xi/d\sigma|_{(1,0)}$ is as claimed.

If $A_\infty^2+B_\infty^2=0$ the formulas for $d\eta/d\sigma|_{(1,0)}$ and $d\xi/d\sigma|_{(1,0)}$ reduce to the correct expressions provided we delete a factor of ξ from each. Q.E.D.

In the previous section the local rate on P_i at points in E^2 was defined. If P_i contains points at infinity the definition of a local rate on P_i may be extended to these points with the aid of Theorem 1.10. Suppose P_i has a multiple point at $(1,0)\varepsilon P_\infty$. Call $\Lambda_{ik}^{\infty}(1,0)$ the local rate on P_i in the Θ_{ik} direction at infinity, and define $\Lambda_{ik}^{\infty}(1,0)$ by

$$\Lambda_{ik}^{\infty}(1,0) = (-\tilde{a}_{ik}^{h})^{c} A_i \prod_{j\neq i}^{n} P_j^{h}(r\tilde{b}_{ik}^{h},-r\tilde{a}_{ik}^{h})^* \times$$

$$\times \left(|P_{i\eta}^{h}(r\tilde{b}_{ik}^{h},-r\tilde{a}_{ik}^{h})|^2+|P_{i\xi}^{h}(r\tilde{b}_{ik}^{h},-r\tilde{a}_{ik}^{h})|^2\right)^{*1/2}, \quad 1.14$$

where $\tilde{a}_{ik}^{h}=a_{ik}^{h}/d$, $\tilde{b}_{ik}^{h}=b_{ik}^{h}/d$, $d=(a_{ik}^{h\,2}+b_{ik}^{h\,2})^{1/2}$, and the kth tangent to P_i^h at $\eta=\xi=0$ is $L_{ik}^{h}=a_{ik}^{h}\eta+b_{ik}^{h}\xi$. Let c=1 for classI flows and c=2 for class II flows. See the general definition of Λ_i in Section C. If the multiple point does not occur at $(1,0)\varepsilon P_\infty$ apply 1.14 to the appropriate rotated flow, that is, replace P_j^h by P_{jR}^h, $j=1,\cdots,n$, in equation 1.14.

Let us now define the local rate on P_∞ at (1,0), $\Lambda_\infty(1,0)$, by

$$\Lambda_\infty(1,0) = \begin{cases} -\left(\sum_{i=1}^{n} A_i n_i\right) \prod_{j=1}^{n} P_j^{h}(r,0)^* & \text{class I} \\ -(rA_\infty+B_\infty)^* \prod_{j=1}^{n} P_j^{h}(r,0)^* & \text{class II} \end{cases} \quad 1.15$$

Using equation 1.13 we may replace $P_j^h(r,0)$ by

$P_{jn_j}(1,r)$. For a class I flow we call $\sum_{i=1}^{n} A_i n_i$ the <u>global rate on P_∞</u> and for a class II flow, $(rA_\infty + B_\infty)^*$ is called the <u>global rate on P_∞</u>. For a class I flow the flow at infinity is completely determined by the flow in E^2, while for a class II flow all but the rate on P_∞ is determined by the flow in E^2.

E. <u>Topological Equivalence of Flows in E^{2+}</u>.

If we wish to study a flow in E^{2+} analytically, we may do so by considering the finite flow in E^2 and, if the flow is algebraic, the flow at infinity in the projective plane. The space CE^{2+} is not an appropriate one in which to study the flow analytically because in CE^{2+} we have three differential equations involved instead of two. However if we wish to study the flow in E^{2+} topologically, neither E^2 nor the projective plane is the proper space because each gives an incomplete picture. In this case we should use CE^{2+}. With these considerations in mind we will base our topological definitions in CE^{2+}.

<u>A vector curve in E^{2+}</u>, designated by $\vec{S}(t)$, is a continuous, differentiable, oriented curve which has associated with each point a nonzero tangent vector pointing in the direction of the orientation. We may

extend this definition to one point curves by calling such curves <u>zero-vector curves</u>, and associating the zero vector with these curves. Every solution to the system $\dot{x}=X(x,y)$, $\dot{y}=Y(x,y)$, provided X and Y satisfy certain minimal regularity conditions, is a vector curve. The critical points correspond to zero-vector curves.

Let the <u>compactified vector curve</u>, written $C\vec{S}(t)$, be the projection of $\vec{S}(t)$ onto CE^{2+}. If we let

$$d_{12}(t) = |C\vec{S}_1(t)-C\vec{S}_2(t)|$$

$$\vec{V}_{12}(t) = C\vec{S}_1(t)/|C\vec{S}_1(t)|-C\vec{S}_2(t)/|C\vec{S}_2(t)|,$$

where $|C\vec{S}_1(t)-C\vec{S}_2(t)|$ is the Euclidean distance between the two points $CS_1(t)$ and $CS_2(t)$ on the unit sphere, and $C\vec{S}(t)/|C\vec{S}(t)|=\vec{0}$ if $\vec{S}(t)$ is a zero-vector curve, then we say that

$$\vec{S}_1^{\pm} \overset{\varepsilon}{\simeq} \vec{S}_2^{\pm} \quad \text{for } \varepsilon>0 \qquad 1.16$$

provided $C\vec{S}_1^{\pm} \overset{\varepsilon}{\simeq} C\vec{S}_2^{\pm}$ in CE^{2+}. The latter relation holds if both $\lim_{t\to\pm\infty} d_{12}(t)<\varepsilon$ and $\lim_{t\to\pm\infty} |\vec{V}_{12}(t)|<\varepsilon$. If $\varepsilon=0$ we say $\vec{S}_1^{\pm} = \vec{S}_2^{\pm}$.

Let us call any open set in E^{2+} which is either connected, or becomes connected when one additional point is added, a <u>region</u>. Two sets of vector curves defined in regions R_1 and R_2 of E^{2+} are <u>o-equivalent in E^{2+}</u> if there exists an orientation preserving homeomorphism of CR_1 onto CR_2 (CR is the image, under the Poincare projection, of R in CE^{2+}) which induces a biunique mapping in CE^{2+} from the first set of vector curves onto the second. A collection of vector curves filling a region R of E^{2+} is called strip parallel if it is o-equivalent to E^2 filled by parallel lines, annularly parallel if it is o-equivalent to $E^2-\{(0,0)\}$ filled by concentric circles, radially parallel if it is o-equivalent to $E^2-\{(0,0)\}$ filled with rays emanating from the origin, and spirally parallel if it is o-equivalent to $E^2-\{(0,0)\}$ filled by spirals emanating from the origin. A collection of vector curves satisfying any one of these four cases is called parallel.

If {S} is a set of vector curves which fills E^{2+} then any two curves, $\vec{S}_1(t)$, $\vec{S}_2(t)\varepsilon\{S\}$, are called ε-parallel if, given any ε>0, there is an integer $M(\varepsilon)<\infty$ and a corresponding finite sequence of vector curves $\vec{T}_1,\cdots,\vec{T}_M\varepsilon\{S\}$ such that; a) $\vec{S}_1^{\pm} \underset{\varepsilon}{=} \vec{T}_1^{\pm} \underset{\varepsilon}{=}\cdots\underset{\varepsilon}{=} \vec{T}_M^{\pm} \underset{\varepsilon}{=} \vec{S}_2^{\pm}$, and b) $\vec{S}_1$, $\vec{T}_1,\cdots$, $\vec{T}_M$, $\vec{S}_2$ belong to the same parallel region for $|t|<\infty$. A vector curve ε-parallel to no others in {S} is called a separatrix and the set of all such vector curves, denoted by $\{\tilde{S}\}$, is called the separatrix system of {S}. Two vector curves, $\vec{S}_1(t)$

and $\vec{S}_1(t)$, not necessarily belonging to the same family of vector curves, are called <u>0-parallel</u> if; a) $\vec{S}_1^{\pm} = \vec{S}_2^{\pm}$, and b) $\vec{S}_1$ and $\vec{S}_2$ are o-equivalent for $|t|<\infty$.

It is clear from the definition that the ε-parallel relation is an equivalence relation on $\{S\}-\{\tilde{S}\}$. This set is then partitioned into disjoint equivalence classes, each class corresponding to a parallel region in the plane bounded by members of $\{\tilde{S}\}$. Let each such region be called a <u>canonical region</u> and call any vector curve belonging to a particular canonical region a <u>canonical representative</u> of that region.

Now consider the family of vector curves associated with a particular system of differential equations $\dot{x}=X(x,y)$, $\dot{y}=Y(x,y)$. In CE^{2+} the separatrix system together with a canonical representative from each canonical region is called the <u>phase portrait</u> of the flow or system. Sometimes the projection of the phase portrait onto either E^2 or the unit disk in the u-v plane, or the projective plane, is also called the "phase portrait" but these projections, taken individually, cannot give a complete picture of the flow.

Since it is difficult to draw the true phase portrait on a two dimensional surface we will use a distortion of the true phase portrait arrived at in the following manner. The flow in the finite plane will be represented by a projection of the corresponding phase portrait onto the unit disk in the u-v plane. At each point on the equator consider the flow in the upper half of the projective plane <u>near</u> the origin and rotate this flow onto the unit disk from above. The vector curves will be somewhat distorted by this rotation but the angle between any vector curve and the η axis will be equal to the angle between its projection and the tangent line to the unit disk.

If two flows have o-equivalent separatrix systems in CE^{2+} and if in corresponding canonical regions the canonical representatives are 0-parallel we say that they are <u>space-equivalent in E^{2+}</u>. Two flows are called <u>time-equivalent in E^{2+}</u> if they become space-equivalent when t is replaced by Kt in one of them, K being a nonzero constant. Flows which are either space-equivalent or time-equivalent in E^{2+} are called <u>topologically equivalent in E^{2+}</u>.

<u>Lemma 1.5</u>: If P_i is replaced by cP_i in the master equation, where $c \neq 0$, then the new flow is time-equivalent in E^{2+} to the old flow.

Proof: Each term in X and Y gains a factor of c if P_i is real, and $|c|^2$ if P_i is complex. Replace t by t/c or $t/|c|^2$. Q.E.D.

We are mainly interested in algebraic systems of differential equations in E^{2+} but the above topological definitions may be applied to any system which satisfies local Lipschitz conditions in regions which do not contain critical points, that is, systems which generate regular curve families. The definitions themselves are primarily extensions of those given by Lawrence Markus [1]. Markus concerned himself exclusively with regular flows in E^2 and considered solution curves rather than vector curves. Hence our definition of ε-parallel vector curves in E^{2+} is stronger than his corresponding definition of parallel curves in E^2. The stronger definitions lead to a finer classification of the algebraic flows, which is also easier to obtain. Usually the topological equivalence of two flows will be obvious from their phase portraits. The definition of "separatrix system" was chosen so that algebraic flows have simple phase portraits. The topological consequences of our definitions and the exact relationships between our definitions and those used by previous authors, will be left to the reader. The following chapters contain many examples which are intended to illustrate and delineate the concepts introduced in this chapter.